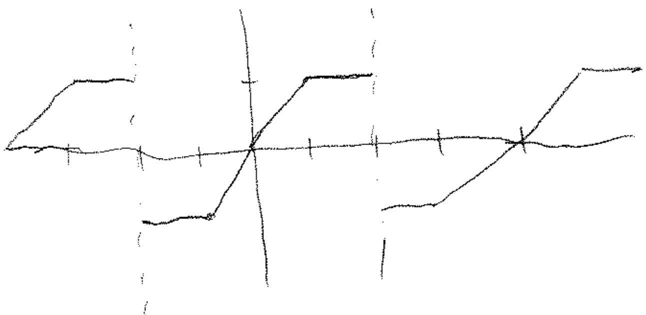
Solutions

1. [10.4.16] Let $f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2 \end{cases}$. Give a sine series for the extension, and sketch the extension to which the series converges.

(1) Sine Series $\Rightarrow$ Need odd extension. $L = 2$.



(2) $f(x) = \frac{a_0}{2} + \sum_{n \ge 1} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$

$a_n = 0$

$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$

$= \frac{2}{2}\left(\int_0^1 x \sin\left(\frac{n\pi}{2}x\right) dx + \int_1^2 \sin\left(\frac{n\pi}{2}x\right) dx\right)$

(3) $\int_0^1 x \sin\left(\frac{n\pi}{2}x\right) dx$

du	dv
x	$\sin\left(\frac{n\pi}{2}x\right)$
1	$-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$
0	$-\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right)$

$= \left(-\frac{2x}{n\pi}\cos\left(\frac{n\pi}{2}x\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}x\right)\right)\Big|_0^1$

$= \left(-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)\right)$

$- \left(-0 + 0\right)$

$= \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)$

$\int_1^2 \sin\left(\frac{n\pi}{2}x\right) dx = \left(-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\right)\Big|_1^2$

$= \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right)$

$= \frac{2(-1)^{n+1}}{n\pi} + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)$

(4) $b_n = \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \frac{2(-1)^{n+1}}{n\pi} \cancel{- \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)}$

(5) $\boxed{f(x) = \sum_{n \ge 1} \left(\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right) + \frac{2(-1)^{n+1}}{n\pi}\right) \sin\left(\frac{n\pi}{L}x\right)}$

2. [10.5.5] If possible, use the method of separation of variables to convert the given partial differential equation to a pair of ordinary differential equations.

   (a) $u_{xx} + (x + y)u_{yy} = 0$ Try $u(x,y) = X(x)\,Y(y)$

$$YX'' + (x+y)XY'' = 0$$

$$YX'' = -(x+y)XY'' \leftarrow \text{Not possible.}$$

   (b) $u_{xx} + u_{yy} + xu = 0$

$$YX'' + XY'' + xXY = 0$$

$$Y(X'' + xX) = -XY''$$

$$\frac{Y''}{Y} = \frac{X'' + xX}{-X} = \lambda \quad \boxed{\begin{array}{l} Y'' = \lambda Y \\ X'' + xX = -\lambda X \end{array}}$$

3. [10.5.12] A rod of 40cm with thermal diffusivity satisfying $\alpha^2 = 1$ has its ends maintained at 0 degrees. Initially, we have $u(x, 0) = x$ for $0 < x < 40$. Determine $u(x, t)$.

$$u(x,t) = \sum_{n\geq 1} c_n u_n(x,t) = \sum_{n\geq 1} c_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$$

$$= \sum_{n\geq 1} c_n \sin\left(\frac{n\pi}{40}x\right) e^{-\left(\frac{n\pi}{40}\right)^2 t}$$

$$c_n = \frac{2}{40}\int_0^{40} x \sin\left(\frac{n\pi}{40}x\right)dx$$

u	dv
x	$\sin\left(\frac{n\pi}{40}x\right)$
1	$-\frac{40}{n\pi}\cos\left(\frac{n\pi}{40}x\right)$
0	$-\left(\frac{40}{n\pi}\right)^2 \sin\left(\frac{n\pi}{40}x\right)$

$$= \frac{2}{40}\left(-\frac{40x}{n\pi}\cos\left(\frac{n\pi}{40}x\right) + \left(\frac{40}{n\pi}\right)^2 \sin\left(\frac{n\pi}{40}x\right)\right)\Bigg|_0^{40}$$

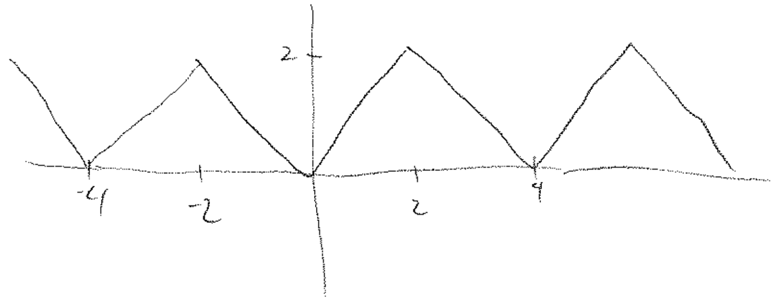
$$= \frac{2}{40}\left(\left(\frac{-(40)^2}{n\pi}\cos(n\pi) + 0\right) - (0 + 0)\right)$$

$$= -\frac{80}{n\pi}(-1)^n$$

$$\text{So } \boxed{u(x,t) = \sum_{n\geq 1} \frac{80}{n\pi}(-1)^{n+1} \sin\left(\frac{n\pi}{40}x\right) e^{-\left(\frac{n\pi}{40}\right)^2 t}}$$

4. [10.4.36] Let $f(x) = x$ for $0 \le x \le 2$ with $f(x+4) = f(x)$.

(a) Sketch an even extension of f.



(b) Find a cosine series for f.

$$f(x) = \frac{a_0}{2} + \sum_{n \ge 1} a_n \cos\left(\frac{n\pi}{L}x\right), \quad L = 2$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$= \int_0^2 x\,dx = \frac{x^2}{2}\Big|_0^2 = \frac{4}{2} - 0 = 2$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi}{2}x\right)dx$$

$$= \int_0^2 x\cos\left(\frac{n\pi}{2}x\right)dx$$

u	dv
x	$\cos\left(\frac{n\pi}{2}x\right)$
1	$+\ \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}x\right)$
0	$-\ -\left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi}{2}x\right)$

$$= \left[\frac{2x}{n\pi}\sin\left(\frac{n\pi}{2}x\right) + \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi}{2}x\right)\right]\Bigg|_0^2$$

$$= \left(4\cdot 0 + \left(\frac{2}{n\pi}\right)^2\cos(n\pi)\right) - \left(0\cdot\sin(0) + \left(\frac{2}{n\pi}\right)^2\cdot 1\right)$$

$$= \left(\frac{2}{n\pi}\right)^2(-1)^n - \left(\frac{2}{n\pi}\right)^2$$

$$= \left(\frac{2}{n\pi}\right)^2\left[(-1)^n - 1\right]$$

$$= \begin{cases} \dfrac{-8}{(n\pi)^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{2}{2} + \sum_{n=1,3,\ldots} \frac{-8}{(n\pi)^2}\cos\left(\frac{n\pi}{2}x\right)$$

$$\boxed{f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1,3,\ldots}\frac{1}{n^2}\cos\left(\frac{n\pi}{2}x\right)}$$

(c) Use part (b) to show that $\frac{\pi^2}{8} = \sum_{n \ge 1}\frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$.

Evaluate at $x = 0$. $f(x) = x$ for $0 \le x \le 2$, so $f(0) = 0$.

From the series:

$$0 = f(0) = 1 - \frac{8}{\pi^2}\sum_{n=1,3,5\ldots}\frac{1}{n^2}\cdot\cancel{\cos(0)}^{\,1} \quad\Big\|\quad \frac{8}{\pi^2}\sum_{n=1,3,5\ldots}\frac{1}{n^2} = 1$$

$$\boxed{\sum_{n=1,3,5}\frac{1}{n^2} = \frac{\pi^2}{8}}$$

5. [10.5.22; Challenging] The heat conduction equation in two space dimensions is $\alpha^2(u_{xx} + u_{yy}) = u_t$. Assuming that $u(x, y, t) = X(x)Y(y)T(t)$, find a system of ordinary differential equations that are satisfied by X, Y, and T.

$$\alpha^2(Y \cdot T \cdot X'' + X \cdot T \cdot Y'') = X \cdot Y \cdot T'$$

$$\alpha^2 T\cancel{XY}\left(\frac{X''}{X} + \frac{Y''}{Y}\right) = \cancel{XY}T'$$

$$\left(\frac{X''}{X} + \frac{Y''}{Y}\right) = \frac{1}{\alpha^2}\frac{T'}{T} = \lambda$$

- $\frac{1}{\alpha^2}\frac{T'}{T} = \lambda \Rightarrow T' = \lambda\alpha^2 T$

- $\frac{X''}{X} + \frac{Y''}{Y} = \lambda$. This requires $\frac{X''}{X}$ and $\frac{Y''}{Y}$ to both be constants that add to λ.

$$\boxed{\frac{X''}{X} = \lambda_1, \quad \frac{Y''}{Y} = \lambda_2, \quad T' = (\lambda_1 + \lambda_2)\alpha^2 T}$$

6. [10.1.8] Either solve the following boundary problem or show that it has no solution: $y'' + 4y = \sin x$ with $y(0) = 0$ and $y(\pi) = 0$.

(1) Solve Homogeneous equ:

$r^2 + 4 = 0$

$r^2 = -4$, $r = \pm 2i$

$y = C_1\cos(2x) + C_2\sin(2x)$

(2) Find particular soln: Undet. Coeff.

$\sin x$
$\downarrow$
$\cos x$
$\downarrow$
~~$-\sin x$~~

$Y(x) = A\sin x + B\cos x$

$Y' = -B\sin x + A\cos x$

$Y'' = -A\sin x - B\cos x$

$y'' + 4y = \sin x$

$(-A + 4A)\sin x + (-B + 4B)\cos x = \sin x$

$3A = 1$, $3B = 0$

$\Rightarrow Y(x) = \frac{1}{3}\sin x$

(3) gen soln:

$y(x) = \frac{1}{3}\sin x + C_1\cos(2x) + C_2\sin(2x)$

(4) Impose Boundary Conditions.

$y(0)=0$: $0 = 0 + C_1 + 0$; $C_1 = 0$

$y(\pi)=0$: $0 = \frac{1}{3}\sin\pi + 0\cdot\cos(2\pi) + C_2\sin(2\pi)$

$0 = 0$ ✓.

(5) soln: $\boxed{y(x) = \frac{1}{3}\sin(x) + C_2\sin(2x)}$.