Name: Solutions

1. **[3 parts, 2 points each]** A spring/mass system is modeled by $6u'' + 7u' + 2u = \cos(\omega t)$.

   (a) Compute the normalized damping constant. Is the system overdamped, critically damped, or neither?

$$\Gamma = \frac{\gamma^2}{mk} = \frac{7^2}{6\cdot 2} = \frac{49}{12} = \frac{48}{12} + \frac{1}{12} = \boxed{4 + \frac{1}{12}}.$$

Since $\Gamma > 4$, the system is $\boxed{\text{overdamped}}$.

   (b) Compute the steady-state response $U(t)$. (Hint: use Cramer's Rule to simplify the process of solving the system of equations.)

- $\gamma > 0 \Rightarrow$ transient sln dies out $\Rightarrow$ no conflict with ~~u~~ family:

$$\cos(\omega t)$$
$$|$$
$$\cancel{\omega}\sin(\omega t)$$
$$|$$
$$\cancel{\omega\cos(\omega t)} \text{ repeat.}$$

- $U(t) = A\cos(\omega t) + B\sin(\omega t)$
- $U' = \omega B\cos(\omega t) - \omega A\sin(\omega t)$
- $U'' = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$

$$6u'' + 7u' + 2u = (-6\omega^2 A + 7\omega B + 2A)\cos(\omega t) + (-6\omega^2 B - 7\omega A + 2B)\sin(\omega t)$$

- $(-6\omega^2 + 2)A + 7\omega B = 1$
- $-7\omega A + (2 - 6\omega^2)B = 0.$

$$\underbrace{\begin{bmatrix} 2-6\omega^2 & 7\omega \\ -7\omega & 2-6\omega^2 \end{bmatrix}}_{M} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\det(M) = (2-6\omega^2)^2 + 49\omega^2 = 4 - \cancel{24}\omega^2 + 36\omega^4 + 49\omega^2 = 36\omega^4 + 25\omega^2 + 4.$$

$$A = \frac{\begin{vmatrix} 1 & 7\omega \\ 0 & 2-6\omega^2 \end{vmatrix}}{\det(M)} = \frac{2-6\omega^2}{36\omega^4 + 25\omega^2 + 4} \qquad B = \frac{\begin{vmatrix} 2-6\omega^2 & 1 \\ -7\omega & 0 \end{vmatrix}}{\det(M)} = \frac{7\omega}{36\omega^4 + 25\omega^2 + 4}$$

   ~~(c) Using part (b), compute the amplitude R of the steady-state in terms of ω.~~

$$U(t) = \frac{2-6\omega^2}{36\omega^4 + 25\omega^2 + 4}\cdot\cos(\omega t) + \frac{7\omega}{36\omega^4 + 25\omega^2 + 4}\sin(\omega t).$$

(c)
$$R = \sqrt{A^2 + B^2} = \left(\frac{(2-6\omega^2)^2 + (7\omega)^2}{(36\omega^4 + 25\omega^2 + 4)^2}\right)^{\frac{1}{2}} = \left(\frac{\cancel{36\omega^4 + 25\omega^2 + 4}}{(36\omega^4 + 25\omega^2 + 4)^{\cancel{2}}}\right)^{\frac{1}{2}}$$

$$= \boxed{\frac{1}{\sqrt{36\omega^4 + 25\omega^2 + 4}}}$$

2. **[2 parts, 2 points each]** *Trig sums.*

(a) Express $\sin(3t) + \sin(4t)$ as the product of two trigonometric functions.

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta)$$

$$\sin(3t) + \sin(4t) = 2\sin\left(\tfrac{7}{2}t\right)\cos\left(-\tfrac{1}{2}t\right)$$
$$= \boxed{2\sin\left(\tfrac{7}{2}t\right)\cos\left(\tfrac{1}{2}t\right)}$$

$$\alpha + \beta = 3t \qquad \alpha + \beta = 3t$$
$$\alpha - \beta = 4t \qquad -(\alpha - \beta = 4t)$$
$$2\alpha = 7t \qquad 2\beta = -t$$
$$\alpha = \tfrac{7}{2}t \qquad \beta = -\tfrac{1}{2}t$$

(b) Determine all real solutions t to $\sin(3t) + \sin(4t) = 0$.

$$2\sin\left(\tfrac{7}{2}t\right)\cos\left(\tfrac{1}{2}t\right) = 0$$

$2 \neq 0$ or $\sin\left(\tfrac{7}{2}t\right) = 0$ or $\cos\left(\tfrac{1}{2}t\right) = 0$

$\tfrac{7}{2}t = k\pi$ for some integer k

$t = \tfrac{2\pi}{7} \cdot k$ for some integer k

$\tfrac{1}{2}t = \pi/2 + k\pi$ for some integer k

$t = \pi + 2\pi k$

$= \pi(2k+1)$ for some integer k

Solns: $t = \ldots, -2 \cdot \tfrac{2\pi}{7}, -1 \cdot \tfrac{2\pi}{7}, 0, 1 \cdot \tfrac{2\pi}{7}, 2 \cdot \tfrac{2\pi}{7}, \ldots$

or $t = \ldots -5\pi, -3\pi, -\pi, \pi, 3\pi, 5\pi, \ldots$