Directions: Solve 5 of the following 6 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. Let G be a nonbipartite triangle-free graph with n vertices and minimum degree k. Let l be the minimum length of an odd cycle in G.

   (a) Let C be a cycle of length l in G. Prove that every vertex not in $V(C)$ contains at most two neighbors in $V(C)$.

   (b) By counting the edges joining $V(C)$ and $V(G) - V(C)$ in two ways, prove that $n \geq kl/2$ (and thus $l \leq 2n/k$).

   (c) When k is even, prove that the inequality of part (b) is best possible. (Hint: form a graph having $k/2$ pairwise disjoint l-cycles.)

2. Determine with proof $\text{ex}(n, P_n)$, the maximum number of edges in an n-vertex graph that does not contain a spanning path.

3. (a) Prove that every connected graph has an orientation in which the number of vertices with odd outdegree is at most 1. (Hint: consider an orientation with the fewest number of vertices with odd outdegree.)

   (b) Use part (a) to conclude that every connected graph with an even number of edges has a P_3-decomposition.

4. [IGT 1.4.29] Suppose that G is a graph and D is an orientation of G that is strongly connected. Prove that if G has an odd cycle, then D has a (directed) odd cycle. (Hint: consider each pair $\{v_i, v_{i+1}\}$ in an odd cycle $v_1 \cdots v_k$ of G.)

5. [IGT 1.4.34] Let G and H be tournaments on the same vertex set. Prove that $d_G^+(v) = d_H^+(v)$ for each vertex v if and only if G can be turned into H by a sequence of direction-reversals on cycles of length 3.

6. Let G be a directed graph without loops. Prove that G has an independent set S such that every vertex in G is reachable from a vertex in S by a directed path of length at most 2. Hint: use induction on $|V(G)|$ and recall that the induction hypothesis applies to all graphs with fewer vertices, not just graphs with $|V(G)| - 1$ vertices.