Directions: Solve the following problems; challenge problems are optional for extra credit. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. [NT 2-2.4] The *least common multiple* of a and b, denoted $\operatorname{lcm}(a, b)$, is the smallest positive integer ℓ such that $a \mid \ell$ and $b \mid \ell$. Prove that if a and b are positive integers, then $\operatorname{lcm}(a, b) = ab/\gcd(a, b)$.

2. [NT 2-3.1] Find the general solution (if solutions exist) of each of the following linear Diophantine equations:

   (a) $15x + 51y = 41$

   (b) $23x + 29y = 25$

   (c) $121x - 88y = 572$

3. Binomial Coefficients and Parity.

   (a) [NT 3-1.3] Using the definition of $\binom{n}{r}$, show combinatorially that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$. (To show an identity combinatorially, find an appropriate set A and show that both sides of the identity count the elements in A.)

   (b) Prove that if n is even and r is odd, then $\binom{n}{r}$ is even.

4. Prove that n^5 and n have the same last digit.

5. Prove that if a and b are positive integers, then it is not possible for both $a + b^2$ and $a^2 + b$ to be square numbers (i.e. of the form k^2 for some integer k). Hint: after a^2, what is the next largest square?

6. A list of consecutive integers $a + 1, \ldots, a + \ell$ is *good* if the squares of its elements sum to a prime number. For example, $5, 6$ is a good list since $5^2 + 6^2 = 61$ and 61 is prime. Let L be the set of all ℓ such that some list of ℓ consecutive integers is good; the example above shows that $2 \in L$. Determine L (with proof). Hint: an identity from HW1 may be useful.

7. **[Challenge]** Prove that if n is an integer and $n \geq 2$, then $n^4 + 4^n$ is not prime.

8. **[Challenge]** Fermat's "medium" theorem?

   (a) Let p and q be distinct primes. Count the number of cyclic lists of length pq with entries in a set of size n. (For example, for $p = 2$, $q = 3$, and $n = 2$, we are counting cyclic lists of length 6 with entries in, say, $\{\text{red}, \text{blue}\}$; there are 14 of these.)

   (b) Use part (a) to show that if p and q are distinct primes and n is a positive integer, then pq divides $n^{pq} - n^p - n^q + n$.