Directions: Solve 6 of the following 7 problems. All written work must be your own, using only permitted sources. See the "General Guidelines and Advice" on the homework page for more details.

1. [IGT 1.2.{17,27}]

   (a) Let G_n be the graph whose vertices are the permutations of $\{1, \ldots, n\}$ with two permutations adjacent if they differ by interchanging a pair of adjacent entries. Note that $G_3 = C_6$. Prove that G_n is connected.

   (b) Let H_n be the graph whose vertices are the permutations of $\{1, \ldots, n\}$ with two permutations adjacent if they differ by interchanging a pair of entries (which may or may not be adjacent). Note that $G_3 = K_{3,3}$ and G_n is a subgraph of H_n. Prove that H_n is bipartite. Hint: for each permutation $a_1 \cdots a_n$, count the pairs (i, j) with $i < j$ and $a_i > a_j$; these are called *inversions*.

2. [IGT 1.2.{29,42}]

   (a) Let G be a connected graph not having P_4 or C_3 as an induced subgraph. Prove that G is a biclique.

   (b) Let G be a connected graph not having P_4 or C_4 as an induced subgraph. Prove that G has a vertex adjacent to all other vertices. (Hint: consider a vertex of maximum degree.)

3. [IGT 1.2.38] Prove that every n-vertex multigraph with at least n edges contains a cycle.

4. [IGT 1.2.40] Let P and Q be paths of maximum length in a connected graph G. Prove that P and Q have a common vertex.

5. A *dominating vertex* in a graph is adjacent to every other vertex. Let G be a graph with no isolated vertices. Prove that $V(G)$ can be partitioned into sets of size at least two such that the subgraph induced by each set has a dominating vertex.

6. [IGT 1.3.{12,18}]

   (a) Prove that an even graph has no cut-edge. For each $k \geq 1$, construct a $(2k+1)$-regular graph having a cut-edge.

   (b) For $k \geq 2$, prove that a k-regular bipartite graph has no cut-edge.

7. [IGT 1.3.20] Count the cycles of length n in K_n and the cycles of length $2n$ in $K_{n,n}$.