Directions: Solve the following problems. Your solutions should be electronically typeset and all written work should be your own.

1. [NT 1-1.5] Prove that $1 + 3 + 5 + \cdots + 2n - 1 = n^2$.
2. [NT 1-1.1] Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
3. Divisibility.
   (a) Prove that if a and n are integers with $a \geq 3$ and $n \geq 2$, then $a^n - 1$ is not prime.
   (b) Prove that if $2^n - 1$ is prime, then n is prime.
4. Give the base 7 representation for 39201.
5. [NT 1-2.{4,5}]
   (a) Find integers $c_0, \ldots, c_s$ with each $c_s \in \{-1, 0, 1\}$ such that
   $$40189 = c_0 + c_1 \cdot 3 + c_2 \cdot 3^2 + \cdots + c_s 3^s.$$
   (b) Prove that each nonzero integer has a unique representation of the form $c_0 + c_1 \cdot 3 + c_2 \cdot 3^2 + \cdots + c_s 3^s$ with each $c_j \in \{-1, 0, 1\}$ and $c_s \neq 0$.
6. Let $d = \gcd(15708, 1870)$. Find d and obtain integers p and q such that $d = 15708p + 1870q$.
7. **[Challenge]** Let $A_n = \{(x, y)\colon\ 1 \leq x \leq n,\ 1 \leq y \leq n, \text{ and } \gcd(x, y) = 1\}$. Note that A_n contains all points (x, y) in the $(n \times n)$-grid with corners $(1, 1)$ and (n, n) such that the line segment joining $(0, 0)$ and (x, y) contains no other integer lattice points. The first few such sets are as follows:
   $$A_1 = \{(1,1)\}$$
   $$A_2 = \{(1,1), (2,1), (1,2)\}$$
   $$A_3 = \{(1,1), (1,2), (1,3), (2,3), (2,1), (3,1), (3,2)\}$$
   Let $f(n) = |A_n|$. Note that $f(1) = 1$, $f(2) = 3$, $f(3) = 7$, and $f(4) = 11$. Prove that there is a positive constant C such that $f(n) \geq Cn^2$.