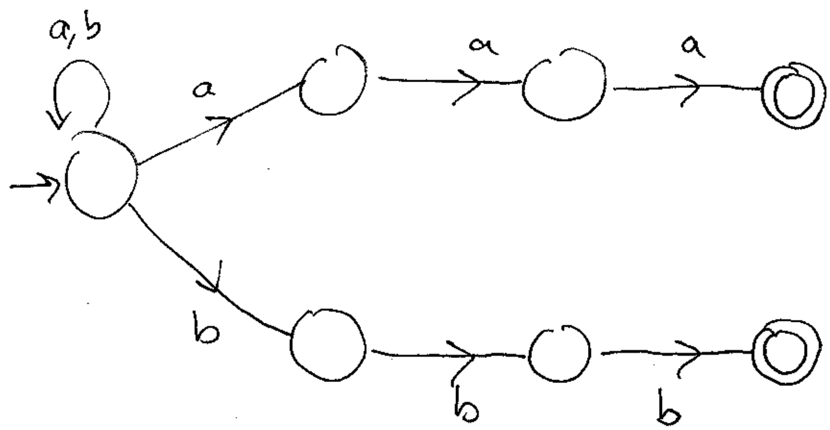
Name: Solutions

Directions: Show all work. Answers without work generally do not earn points.

1. **[10 points]** Let $\Sigma = \{a, b\}$. Construct a simple NFA for the language $\{w \mid w \text{ ends with } aaa \text{ or } bbb\}$.



2. **[3 parts, 2 points each]** Let $\Sigma = \{a, b\}$. Let

$$A = \{w \mid w \text{ has at least as many } a\text{'s as } b\text{'s}\}$$
$$B = \{w \mid w \text{ has at least as many } b\text{'s as } a\text{'s}\}.$$

For each language below, give a DFA if the language is regular or write "not regular" if the language is not regular.

(a) A

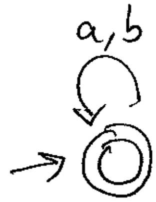
not regular

(b) B

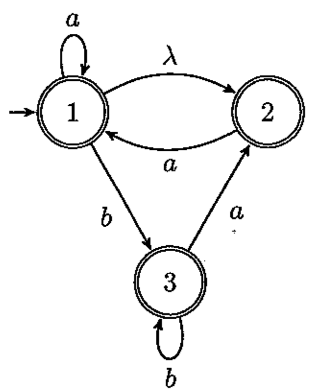
not regular

(c) AB

regular: $AB = \{$ set of all strings $\}$

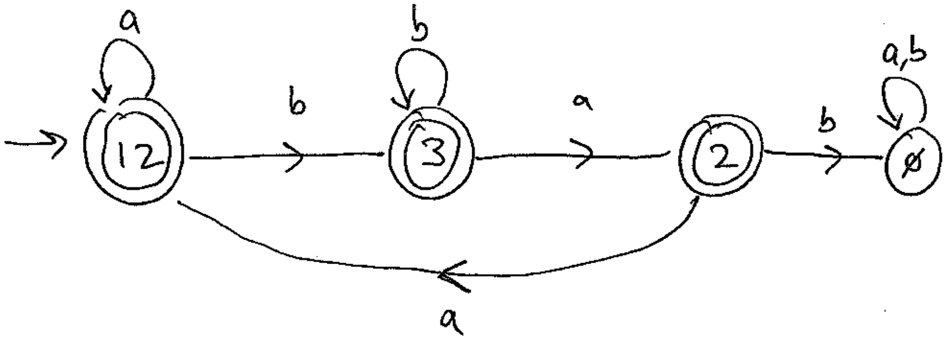


3. Let $\Sigma = \{a, b\}$ and let N be the following NFA.



(a) **[10 points]** Convert N to an equivalent DFA.

	a	b
1	12	3
2	12	$\emptyset$
3	2	3



(b) **[2 points]** What is the shortest word that is *not* accepted by N? $\boxed{bab}$

(c) **[2 points]** Give a simple, English description of $L(N)$.

L(N) is the set of all strings that do not contain bab as a substring.

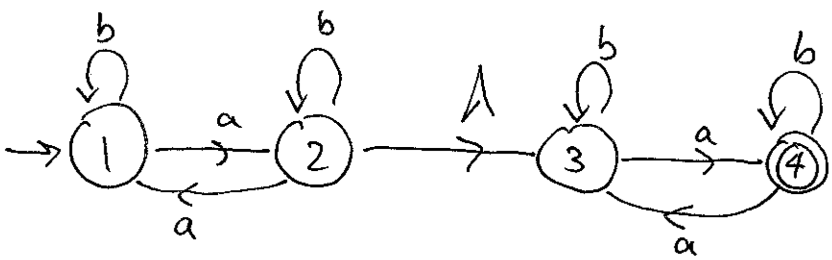
4. **[5 points]** Let N be an NFA that recognizes a language A. Describe how to use N to make a new NFA or DFA which recognizes the complement language $\{w \mid w \notin A\}$.

Important Step $\Rightarrow$ First, convert N to an equivalent DFA M.

Next, switch all accepting states in M to rejecting states, and switch all rejecting states to accepting states. The language of the resulting DFA is $\{w \mid w \notin A\}$.

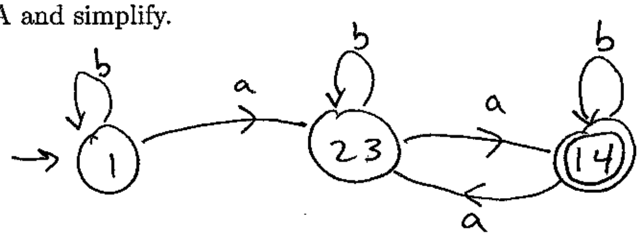
5. **[2 parts, 5 points each]** Let $\Sigma = \{a, b\}$, and let $A = \{w \mid w$ has an odd number of a's$\}$.

(a) Give an NFA for the language AA.



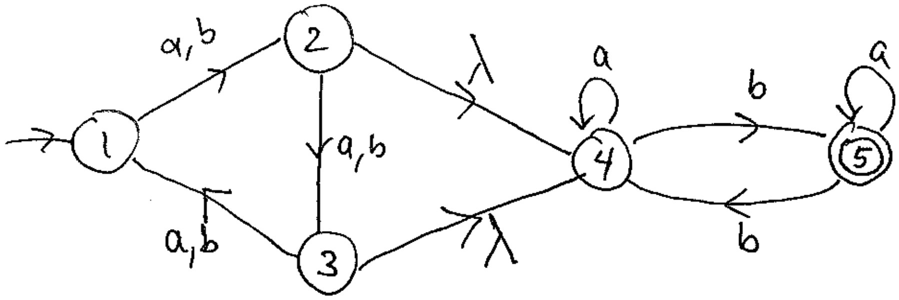
(b) Convert the NFA to a DFA and simplify.

	a	b
1	23	1
2	14	23
3	4	3
4	3	4



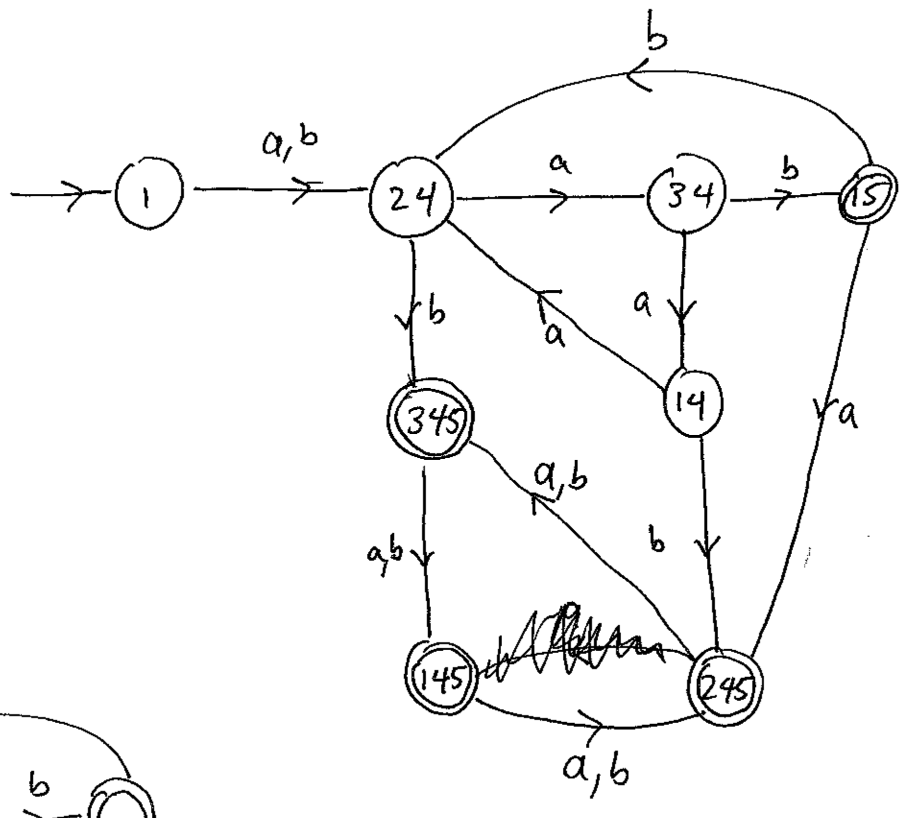
6. **[2 parts, 8 points each]** Let $\Sigma = \{a, b\}$, and let $A = \{w \mid \text{the length of } w \text{ is not a multiple of } 3\}$, and let $B = \{w \mid w \text{ has an odd number of } b\text{'s}\}$.

(a) Give an NFA for the language AB.

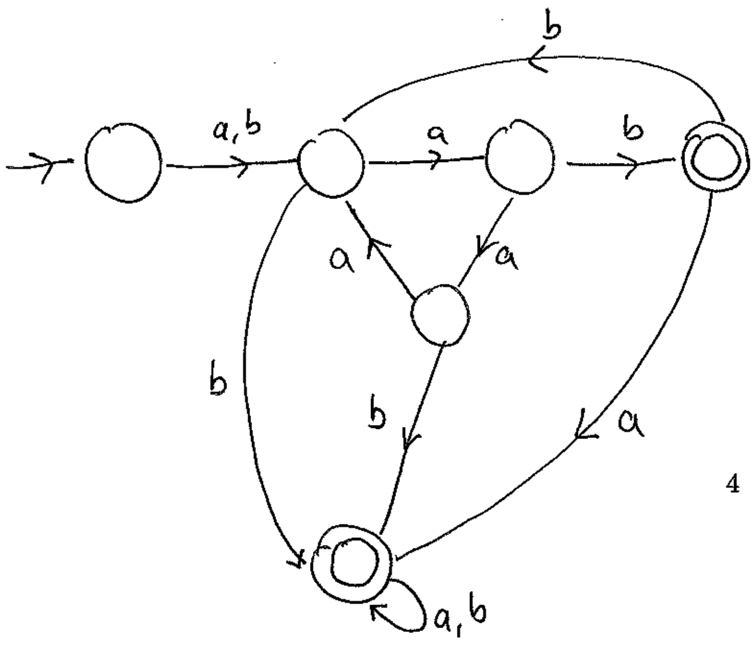


(b) Convert the NFA to a DFA and simplify.

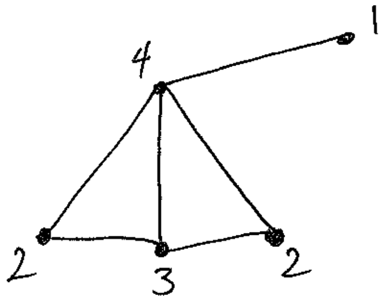
	a	b
1	24	24
2	34	345
3	14	15
4	4	5
5	5	4



Simplified:



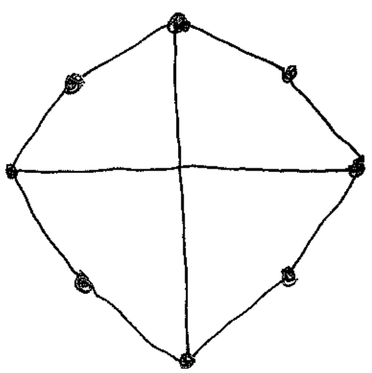
7. **[5 points]** Draw a 5-vertex graph (without loops and without parallel edges) in which one vertex has degree 4, one vertex has degree 3, two vertices have degree 2, and one vertex has degree 1.



8. **[5 points]** Give a simple argument that there is no 15-vertex graph (without loops and without parallel edges) in which two vertices have degree 14 and one vertex has degree 1.

In a 15-vertex graph, a vertex with degree 14 is adjacent to every other vertex. If two vertices, u and v, both have degree 14, then every other vertex is adjacent to u and v and therefore has degree at least 2. So no vertex has degree 1.

9. **[5 points]** Find an 8-vertex graph which contains C_5, C_6, and C_8 as subgraphs but does not contain cycles of any other lengths.

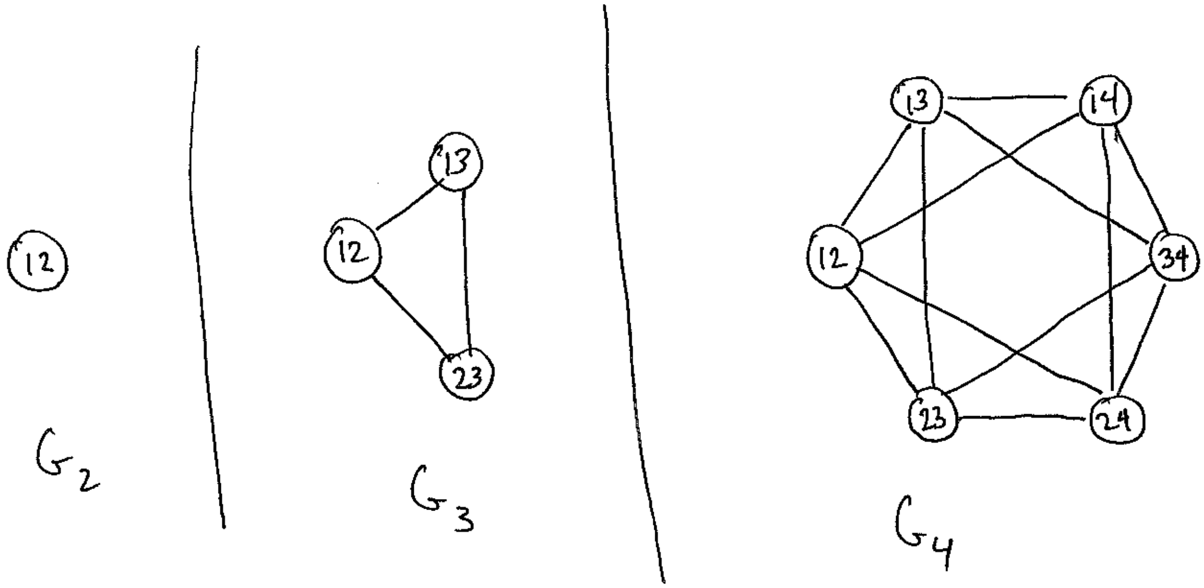


Explanation:

1. First draw C_8.
2. Every other edge must join vertices at distance 4 along C_8 (or else we get a C_3 or C_4)
3. Consecutive chords would give a C_4.

10. For $n \geq 2$, let G_n be the graph whose vertices are the subsets of $\{1, 2, \ldots, n\}$ of size 2 and where two vertices are adjacent if and only if they have non-empty intersection. For example, in G_5, the vertices $\{3,4\}$ and $\{3,5\}$ are adjacent but $\{3,4\}$ and $\{1,5\}$ are not adjacent.

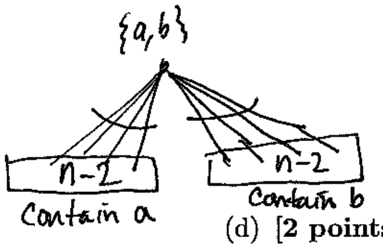
(a) **[6 points]** Draw G_2, G_3, and G_4.



(b) **[2 points]** Give a formula for the number of vertices in G_n.

$$\boxed{\binom{n}{2}}$$

(c) **[2 points]** Give a formula for the degree of a vertex in the regular graph G_n.



$$\boxed{2(n-2)}$$

(d) **[2 points]** Use parts (b) and (c) to give a formula for the number of edges in G_n.

$$|E(G_n)| = \frac{1}{2} \sum_{v \in V(G_n)} d(v) = \frac{1}{2} \cdot |V(G)| \cdot 2(n-2)$$

$$= \binom{n}{2} \cdot (n-2) = \boxed{\frac{n(n-1)(n-2)}{2}}$$

11. [**3 parts, 4 points each**] For each pair of graphs below, decide if the graphs are isomorphic. If they are isomorphic, give a table. If not, argue why not.

(a)

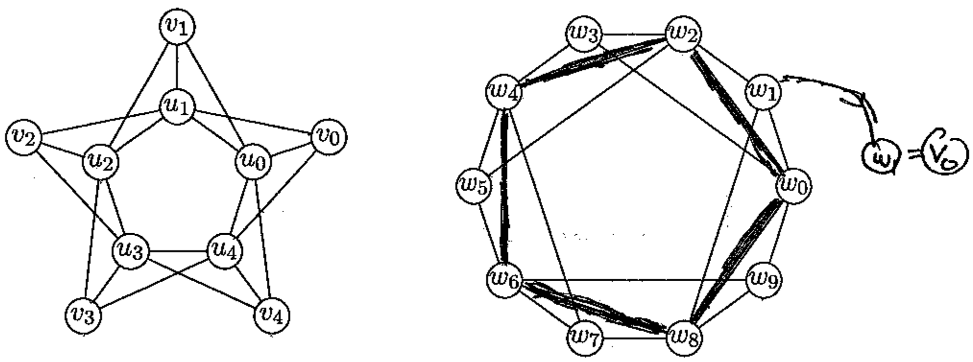
Isomorphic:

u_1	u_2	u_3	u_4
v_2	v_1	v_3	v_4

(b)

Not isomorphic: Graph on left contains triangles; graph on right does not.

(c)



Inner 5-cycle in bold

Isomorphic: Inner cycle

u_0	u_1	u_2	u_3	u_4
w_0	w_2	w_4	w_6	w_8

Outer verts:

v_0	v_1	v_2	v_3	v_4
w_1	w_3	w_5	w_7	w_9