Name: ______________________________

Directions: Show all work. Answers without work generally do not earn points.

1. **[3 parts, 3 points each]** Let $A = \{-9, 2, 4, \{5, 6\}, (9, 7), \varnothing\}$, let $B = \{n^2 \mid n \in \mathbb{Z}\}$, let $C = \{4, \{6, 5\}, (7, 9), \{\varnothing\}\}$.

   (a) Determine $|A|$ and $|C|$.

   (b) Find $A \cap C$. What is $|A \cap C|$?

   (c) Find $A \cap B$. What is $|A \cap B|$?

2. **[5 points]** Let $A = \{1\}$. Find $\mathcal{P}(\mathcal{P}(A))$.

3. **[5 points]** Let A be the set of all subsets of $\{1, 2, 3, 4, 5\}$ that do not contain two consecutive integers. List the elements of A. What is $|A|$?

4. **[5 parts, 3 points each]** A town of n people needs to form a committee of k people with a leader. (The leader must be one of the committee members.)

   (a) Suppose we choose the k committee members first and then we choose a leader from the committee. Using this scheme, determine the number of ways to select a committee of size k with a leader.

   (b) Suppose we choose the leader first and then choose the rest of the committee. Using this scheme, determine the number of ways to select a committee of size k with a leader.

   (c) What conclusion, if any, can you draw from parts (a) and (b)?

   (d) The town decides the size of the committee is no longer important. Using the scheme where the leader is chosen first, count the number of ways to form a committee *of any size* with a leader.

   (e) Find a simple formula for the sum $\sum_{k=0}^{n} k\binom{n}{k}$.

5. **[3 parts, 4 points each]** Recall that $\mathbb{N} \times \mathbb{N} = \{(x, y) \mid x \in \mathbb{N} \text{ and } y \in \mathbb{N}\}$. Each of the following parts claims to list the elements of $\mathbb{N} \times \mathbb{N}$ (and therefore prove that $\mathbb{N} \times \mathbb{N}$ is countable). Decide whether or not each list is correct. If incorrect, describe why.

   (a) Begin by listing the pairs where $y = 0$, so that the list begins $(0, 0), (1, 0), (2, 0), \ldots$. Next, list all the pairs where $y = 1$, so that the list continues $(0, 1), (1, 1), (2, 1), \ldots$. Next list all the pairs where $y = 2$, and so on.

   (b) Begin by listing all the pairs (x, y) where $\max(x, y) = 0$, so that the list begins $(0, 0)$. Next, list all the pairs where $\max(x, y) = 1$, so that the list continues $(1, 0)$, $(1, 1)$, $(0, 1)$. Next, list all the pairs where $\max(x, y) = 2$, and so on.

   (c) For each possible value of x, we iterate over all values of y from 0 to x. The list begins $(0, 0), (1, 0), (1, 1), (2, 0), (2, 1), (2, 2), (3, 0), \ldots$.

6. **[4 points]** Why did mathematicians switch from Naive Set Theory to Axiomatic Set Theory?

7. Let $\Sigma = \{0, 1\}$.

   (a) **[3 points]** What is $|\Sigma^3|$?

   (b) **[3 points]** Write down the set Σ^0 explicitly.

   (c) **[4 points]** Let A be the set of all strings over Σ of even length and let B be the set of all strings over Σ of odd length. Give a simple English description for the language AB.

8. **[2 parts, 4 points each]** Let $\Sigma = \{a, b, c\}$. Let D be the set of all words over Σ in which every a appears before every b. For example, *aacaccacbbb* and *bbcb* are both in D but *bbab* is not in D. Let E be the set of all words over Σ in which every b appears before every a.

   (a) Is it true that $D \cup E = \Sigma^*$? Explain why or why not.

   (b) Give a simple, English description for the language $D \cap E$.

9. **[3 parts, 4 points each]** Let $\Sigma = \{0, 1\}$ and let A be the language over Σ defined recursively as follows:

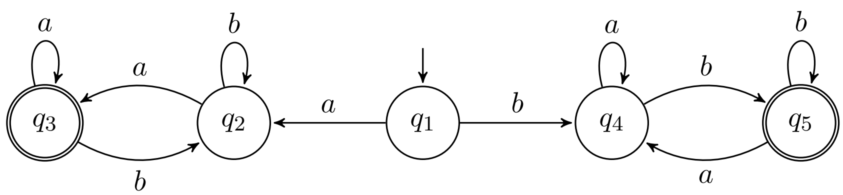
   1. $\lambda \in A$
   2. If $x \in A$, then $x0x \in A$.
   3. If $x \in A$, then $x1x \in A$.

   (a) List all words in A of length at most 3.

   (b) How many words in A have length 7?

   (c) Give an example of a word of length 7 that is a palindrome but is not in A.

10. Let $\Sigma = \{a, b\}$ and let M be the following automaton.



   (a) **[4 points]** List the sequence of states that results when M is given $abbab$ as input. Is $abbab \in L(M)$?

   (b) **[6 points]** Give a simple English description of $L(M)$.

11. **[10 points]** Let $\Sigma = \{a, b, c\}$, and let A be the language of all strings over Σ that do not contain consecutive repeated symbols. For example, $abacbcba \in A$ but $abbac \notin A$. Construct a finite automaton that recognizes A.