Name: Solutions

Directions: Show all work. Answers without work generally do not earn points. Unless stated otherwise, answers may be left in terms of factorials and binomial coefficients.

1. [**4 points**] A restaurant offers 7 different sandwiches, 5 sides, 6 soups, and 3 desserts. The lunch special consists of a sandwich, a choice of 1 side or a soup (but not both), and a dessert. How many ways are there to order the lunch special?

   1. Choose sandwich $(n_1 = 7)$
   2. Choose soup or side $(n_2 = 5+6)$
   3. Choose dessert $(n_3 = 3)$.

   $7 \cdot 11 \cdot 3 = \boxed{231}$

2. [**4 points**] How many ways are there to distribute 35 identical gold coins among 8 people?

   35 stars
   7 bars

   $\binom{35+7}{7} = \binom{42}{7} = \boxed{26{,}978{,}328}$

3. [**2 parts, 4 points each**] How many 5-digit ATM pins:

   (a) contain only even digits?

   Only 0, 2, 4, 6, 8.

   $\Rightarrow 5^5 = 3125$

   (b) contain at least one odd digit?

   All (# pins) = (# pins with only even digits) + x

   $10^5 = 3125 + x$

   $x = 10^5 - 5^5 = \boxed{96{,}875}$

4. **[3 parts, 4 points each]** A game system has 4 buttons in different colors: red, green, blue, and yellow. The buttons must be pressed one at a time, in some order. To win the game, each button must be pressed twice. How many ways are there to win:

   (a) with no additional restrictions?

Count arrangements of RRGGBBYY.

$$= \frac{8!}{(2!)(2!)(2!)(2!)} = \frac{8!}{2^4} = \boxed{2,520}$$

   (b) if the green presses must occur consecutively?

Count arrangements of $RR\underbrace{\langle GG \rangle}_{}^{\text{single symbol}}BBYY$.

$$= \frac{7!}{2!\cdot 2!\cdot 2!} = \frac{7!}{2^3} = \boxed{630}$$

   (c) if both red presses must occur before both blue presses?

⇒ Replace R and B with ?; arrange; replace ?'s with RRBB.

Count arrangements of ??GG??YY

$$\frac{8!}{(4!)(2!)(2!)} = \boxed{420}.$$

5. **[3 parts, 4 points each]** *Word arrangements.* How many ways are there to arrange the letters of 'APPROPRIATE':

   (a) with no additional restrictions.

A - 2
P - 3
R - 2
O - 1
I - 1
T - 1
E - 1

$$\frac{(11)!}{(2!)(3!)(2!)} = \boxed{1,663,200}$$

   (b) with no two consecutive P's.

① Arrange A R O R I A T E

$$\frac{8!}{(2!)(2!)} = 10,080$$

② Insert P's $\binom{9}{3}$

$$\text{Total \# ways} = \binom{9}{3} \cdot \frac{8!}{(2!)(2!)} = 84 \cdot 10,080 = \boxed{846,720}$$

   (c) with all P's separated by at least 2 letters. (So PAAPROPRITE counts but PAPROPRIATE does not.

① Write down P's: P P P

x_1 x_2 x_3 x_4

$n_1 = 1$

② Decide how many letters go in the spaces around P's. #ways to do this is the # of non-neg. integer solns to $x_1+x_2+x_3+x_4 = 8$, with $x_2 \geq 2$ and $x_3 \geq 2$.

$n_2 = \binom{7}{3}$

③ Arrange other letters in some order

$n_3 = \frac{8!}{(2!)(2!)}$

Ans: $\binom{7}{3} \frac{8!}{(2!)(2!)} = \boxed{352,800}$

6. *Poker hands.* Recall that a deck of cards has 4 suits (clubs, diamonds, hearts, and spades) and 13 ranks (ace, 2 through 10, jack, queen, and king). There are 52 cards (one for each suit/rank pair). A poker hand is a set of 5 cards (order does not matter). The *face cards* are the cards whose rank is jack, queen, or king.

(a) **[4 points]** How many poker hands have no face cards?

$\#$ Face cards $= 3 \cdot 4 = 12$

$\#$ non-face cards $= 52 - 12 = 40$.

$\#$hands no face cards $= \binom{40}{5} = \cancel{6861} = \boxed{658{,}008}$

(b) **[1 point]** What are the odds of being dealt a poker hand with no face cards? Round your answer to the nearest decimal percentage of the form $xx.xx\%$.

$$\text{odds} = \frac{658{,}008}{\binom{52}{5}} = \frac{658{,}008}{2{,}598{,}960} \approx 0.2532 \quad \text{so } \boxed{25.32\%}$$

(c) **[4 points]** How many hands have 3 cards in one suit and 2 cards in a different suit?

1. Choose dominant suit $n_1 = 4$
2. Choose 3 ranks for dom. suit $n_2 = \binom{13}{3}$
3. Choose minor suit differently $n_3 = 3$
4. Choose 2 ranks for minor suit $n_4 = \binom{13}{2}$.

$$\text{Total} = 4\binom{13}{3} 3 \binom{13}{2} = \boxed{267{,}696}$$

(d) **[4 points]** How many hands have all distinct ranks and at least 1 card in each suit?

~~1. Choose 5 distinct ranks~~ $\cancel{n_1 = \binom{13}{5}}$

~~2. Choose a pair of cards that will have the same suit~~ $\cancel{n_2 = \binom{5}{2}}$

Better:

1. Choose a suit which will have 2 cards $n_1 = 4$
2. Choose ranks for the doubled suit $n_2 = \binom{13}{2}$
3. Choose ~~a rank~~ different ranks in the 3 remaining suits $n_3 = 11 \cdot 10 \cdot 9$.

$$\text{Total} = 4\binom{13}{2} \cdot 11 \cdot 10 \cdot 9 = \boxed{308{,}880}$$

7. **[5 parts, 3 points each]** Count the non-negative integer solutions to $x_1 + \cdots + x_5 = 40$:

(a) with no additional restrictions.

$$\left.\begin{array}{l}40 \text{ stars} \\ 4 \text{ bars}\end{array}\right\} \binom{40+4}{4} = \binom{44}{4} = \boxed{135,751}$$

(b) with $x_i \geq 3$ for all i.

$$y_1 + \cdots + y_5 = 40 - 3\cdot 5$$

$$y_1 + \cdots + y_5 = 25$$

$$\left.\begin{array}{l}25 \text{ stars} \\ 4 \text{ bars}\end{array}\right\} \Rightarrow \binom{29}{4} = \boxed{23,751}$$

(c) with $x_2 \leq 18$

with $x_2 \geq 19$:

$$y_1 + \cdots + y_5 = 21$$

$$\#\text{ solns} = \binom{21+4}{4} = \binom{25}{4}$$

(Total # solns) = (# with $x_2 \leq 18$) + (# with $x_2 \geq$

$$\binom{44}{4} = x + \binom{25}{4}$$

$$x = \binom{44}{4} - \binom{25}{4} = \boxed{123,101}$$

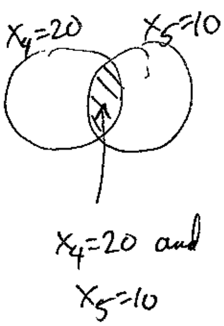
(d) with $x_4 = 20$ and $x_5 = 10$

$$x_1 + x_2 + x_3 + 20 + 10 = 40$$

$$x_1 + x_2 + x_3 = 10$$

$$\binom{10+2}{2} = \binom{12}{2} = \boxed{66}$$

(e) with $x_4 = 20$ or $x_5 = 10$ (or both)



with $x_4 = 20$:

$$x_1 + x_2 + x_3 + x_5 = 20$$

$$\binom{23}{3}$$

with $x_5 = 10$:

$$x_1 + x_2 + x_3 + x_4 = 30$$

$$\binom{33}{3}$$

So

$$x = \binom{23}{3} + \binom{33}{3} - \binom{12}{2}$$

$$= \boxed{7,161}$$

8. **[3 parts, 4 points each]** Find the coefficient:

(a) of x^4y^5 in $(x+y)^9$

$$\binom{9}{4} = \frac{9!}{(4!)(5!)} = \boxed{126}$$

(b) of $x^3y^4z^5$ in $(x+y+z)^{12}$

$$\frac{(12)!}{(3!)(4!)(5!)} = \boxed{27,720}$$

(c) of $x^5y^5z^5$ in $(2x-y+3z)^{15}$

In $(A+B+C)^{15}$, $A^5B^5C^5$ has coefficient $\frac{(15)!}{(5!)(5!)(5!)}$

So $\frac{(15)!}{(5!)(5!)(5!)} \cdot 2^5 \cdot (-1)^5 \cdot (3)^5 x^5y^5z^5 = \boxed{-6^5 \cdot \frac{15!}{(5!)^3}}$

$\boxed{-5,884,534,656}$

9. **[2 parts, 4 points each]** Give simple formulas for the following sums:

(a) $\sum_{k=1}^{n} k$

$$= 1 + 2 + \cdots + n = \boxed{\frac{n(n+1)}{2}} = \boxed{\binom{n}{2}}$$

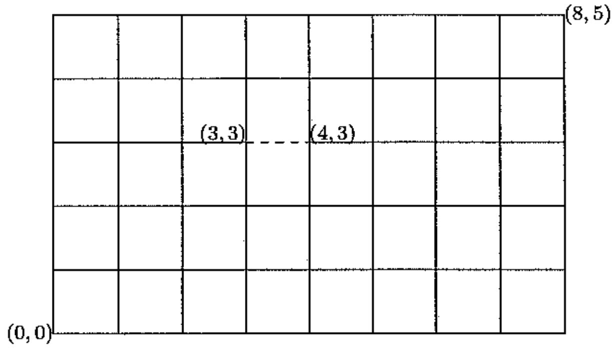
Gauss

(b) $\sum_{k=0}^{n} \binom{n}{k} 3^k$

$$= \sum_{k=0}^{n} \binom{n}{k} 3^k (1)^{n-k} = (3+1)^n = \boxed{4^n}$$

Binomial Theorem

10. **[2 parts, 4 points each]** *Lattice Paths.* Recall that a step in a lattice path increases one of the coordinates by 1.



(a) How many lattice paths are there from $(0, 0)$ to $(8, 5)$?

$$\frac{(8+5)!}{(5!)(8!)} = \binom{13}{5} = \boxed{1287}$$

(b) How many of these paths avoid the segment from $(3, 3)$ to $(4, 3)$ (depicted above with a dashed line segment)?

$(0,0)$ to $(3,3)$: $\frac{6!}{(3!)(3!)} = \binom{6}{3}$

$(4,3)$ to $(8,5)$: $\frac{(4+2)!}{(4!)(2!)} = \binom{6}{2}$

~~#~~total paths = #paths that cross + X

$$\binom{13}{5} = \binom{6}{3}\binom{6}{2} + X$$

$$X = \binom{13}{5} - \binom{6}{3}\binom{6}{2} = \boxed{987}$$

11. **[4 points]** *Lattice paths in 3 dimensions.* In 3 dimensions, a step in a lattice path moves from (x, y, z) to one of the following points: $(x+1, y, z)$, $(x, y+1, z)$, $(x, y, z+1)$. How many lattice paths are there from $(0, 0, 0)$ to (n, n, n)? Hint: apply the method that allowed us to count 2-dimensional lattice paths.

Arrange $\underbrace{XX\cdots X}_{n \text{ copies}}$ $\underbrace{YY\cdots Y}_{n \text{ copies}}$ $\underbrace{ZZ\cdots Z}_{n \text{ copies}}$ in some order.

$$\frac{(n+n+n)!}{(n!)(n!)(n!)} = \boxed{\frac{(3n)!}{(n!)^3}}$$