Name: Solutions

Directions: Show all work. No credit for answers without work.

1. **[3 points]** Solve for y explicitly: $\frac{dy}{dx} = \frac{\sin(2x)}{2y}$ with $y(0) = -1$.

$$\int 2y\,dy = \int \sin(2x)\,dx$$

$$y^2 = -\frac{1}{2}\cos(2x) + C$$

$\underline{y(0) = -1}$:

$$(-1)^2 = -\frac{1}{2}\cancel{\cos}(0) + C$$

$$\frac{3}{2} = C$$

$$y^2 = -\frac{1}{2}\cos(2x) + \frac{3}{2}$$

$$y = \pm\sqrt{\frac{3}{2} - \frac{1}{2}\cos(2x)}$$

Since $y(0)=-1$, choose the negative branch:

$$\boxed{y = -\sqrt{\frac{3}{2} - \frac{1}{2}\cos(2x)}}$$

2. **[3 points]** Solve for y: $\frac{dy}{dx} = \frac{y^3}{x^3 + y^2 x}$ with $y(1) = 1$. Implicit solutions are permitted.

$$\frac{dy}{dx} = \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \cdot \frac{y^3}{x^3 + y^2 x} = \frac{\left(\frac{y}{x}\right)^3}{1 + \left(\frac{y}{x}\right)^2}$$

$$v = \frac{y}{x}, \quad y = vx, \quad \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$x\frac{dv}{dx} + v = \frac{v^3}{1+v^2}$$

$$x\frac{dv}{dx} = \frac{v^3}{1+v^2} - v$$

$$x\frac{dv}{dx} = \frac{v^3 - (v + v^3)}{1+v^2}$$

$$x\frac{dv}{dx} = -\frac{v}{1+v^2}$$

Note: this gives the soln $y=0$

$$\frac{1+v^2}{v}\,dv = -\frac{1}{x}\,dx \qquad (\text{or } v=0)$$

$$\int \frac{1}{v}\,dv + \int v\,dv = -\int \frac{1}{x}\,dx$$

$$\ln|v| + \frac{v^2}{2} = -\ln|x| + C$$

$$\ln\left|\frac{y}{x}\right| + \ln|x| + \frac{y^2}{2x^2} = C$$

$$\ln|y| + \frac{y^2}{2x^2} = C$$

$\underline{\text{Impose } y(1)=1}$:

$$\ln|1| + \frac{1}{2} = C; \quad C = \frac{1}{2}$$

$$\boxed{\begin{aligned} \ln|y| + \frac{y^2}{2x^2} &= \frac{1}{2} \\ \ln(y^2) + \frac{y^2}{x^2} &= 1 \end{aligned}}$$

3. Suppose that $y' = 4(1+2x)(1+y^2)$ with $y(0) = 0$.

   (a) **[2 points]** Solve the IVP.

$$\int \frac{1}{1+y^2}\,dy = \int 4(1+2x)\,dx$$

$$\arctan(y) = (1+2x)^2 + C$$

Impose $y(0)=0$:

$$\arctan(0) = 1 + C$$

$$0 = 1 + C$$

$$C = -1$$

$$\arctan(y) = (1+2x)^2 - 1$$

$$\cancel{y = \tan\left(2(1+x)^2 - 2\right)}$$

$$\cancel{y = \tan(1+2x)^2 - 1}$$

$$\boxed{y = \tan\left((1+2x)^2 - 1\right)}$$

valid only when

$$\left|(1+2x)^2 - 1\right| < \frac{\pi}{2}$$

or approximately in the interval $(-1.30, 0.30)$

   (b) **[2 points]** Determine where the solution attains its minimum value.

Since $y' = 4(1+2x)(1+y^2)$, and $(1+y^2)$ is always positive,

$y' > 0$ when $1+2x > 0$ or $x > -1/2$

$y' < 0$ when $1+2x < 0$ or $x < -1/2$

So, $y(x)$ decreases when $x < -1/2$ and increases when $x > -1/2$.

Therefore the minimum occurs at $\boxed{x = -\frac{1}{2}}$.