Directions: Solve the following 6 problems. See the course syllabus and the Homework Webpage on the course website for general directions and guidelines.

1. Partition Exercises.
   (a) Find the conjugate partition to $16 = 5 + 4 + 4 + 2 + 1$.
   (b) [NT 12-3.1] For the case $n = 8$, list the corresponding pairs of partitions of n in which all parts are odd and partitions of n into distinct parts given by Theorem 12-3.
2. Sums of three squares.
   (a) [NT 11-2.9] Show that no integer of the form $4^a(8m + 7)$ is the sum of three squares. Hint: consider the congruence $x^2 + y^2 + z^2 \equiv 7 \pmod 8$.
   (b) Prove or disprove: if x and y are representable as the sum of three squares, then so is xy.
3. Let p be a prime.
   (a) Let a be an integer such that $p \nmid a$, and let h be the order of a. Show that if $a \not\equiv 1 \pmod p$, then $1 + a + a^2 + \cdots + a^{h-1} \equiv 0 \pmod p$.
   (b) Let $Q = \{a\colon\ 1 \le a \le p - 1 \text{ and } a \text{ is a quadratic residue}\}$. Prove that if $p \ge 5$, then $\sum_{t\in Q} t \equiv 0 \pmod p$.
   (c) [**Challenge**] Let $R = \{a\colon\ 1 \le a \le p - 1 \text{ and } a \text{ is a primitive root}\}$. Prove that $\sum_{t\in R} t \equiv \mu(p - 1) \pmod p$, where $\mu(n)$ is the Möbius function.
4. Prove that the only integral solutions to $2^a - 3^b = 1$ are $(a, b) = (1, 0)$ and $(a, b) = (2, 1)$. Hint: look at the equation modulo 3 and modulo 4.
5. Let p be an odd prime. Determine the number of mutually incongruent solutions to $x^2+y^2 \equiv 0 \pmod p$. (A solution (x, y) is congruent to (x', y') if $(x, y) \equiv (x', y') \pmod p$. When $p = 3$, there is 1 solution $(0, 0)$, and when $p = 5$, there are 9 solutions.)
6. Let $P(q)$ be the generating function for the partition numbers. That is, $P(q) = \sum_{n\ge 0} p(n)q^n$ by definition, and $P(q) = \prod_{j\ge 1} \frac{1}{1-x^j}$ for $|q| < 1$ by Theorem 13–3.
   (a) Let $a_k(n)$ be the number of partitions of n in which each part is used less than k times, and let $A_k(q)$ be the generating function $A_k(q) = \sum_{n\ge 0} a_k(n)q^n$. Show that $A_k(q) = \frac{P(q)}{P(q^k)}$ for $|q| < 1$.
   (b) Let $b_k(n)$ be the number of partitions of n in which no part is divisible by k, and let $B_k(q)$ be the generating function $B_k(q) = \sum_{n\ge 0} b_k(n)q^n$. Show that $B_k(q) = \frac{P(q)}{P(q^k)}$ for $|q| < 1$.

   Note: Since $A_k(q) = B_k(q)$, it follows that $a_k(n) = b_k(n)$ for all n.