Name: Solutions

Directions: Show all work. No credit for answers without work. This test has 100 points but scores will be taken out of 88.

1. **[12 points]** Let $f(x, y) = \ln(y/x^2)$. (i) What is the domain of f? (ii) Sketch a contour map of f showing ~~four~~ the level curves of heights 1 and 2. Label each curve with its height.

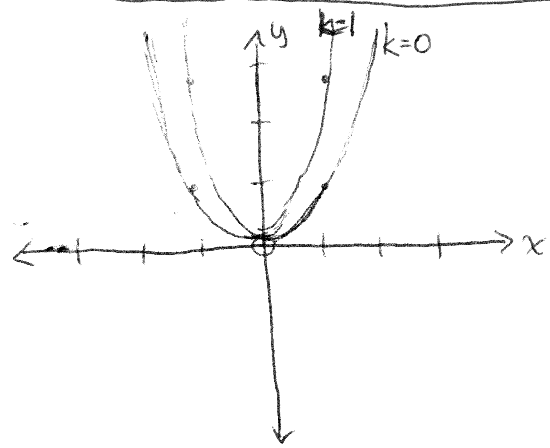
(i) Need $\frac{y}{x^2} > 0$, so $y > 0$ and $x \neq 0$. $\boxed{D = \{(x,y) : x \neq 0 \text{ and } y > 0\}}$

(ii) $\ln\left(\frac{y}{x^2}\right) = k$

$$\frac{y}{x^2} = e^k$$

$$y = e^k x^2$$

parabolas with vertex $(0,0)$



2. **[2 parts, 6 points each]** Find the limit, if it exists, or show that the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + xy}{x^2 + y^2}$

$x = 0$: $\displaystyle\lim_{y\to 0} \frac{0^2 + 0y}{0^2 + y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$

$y = 0$: $\displaystyle\lim_{x\to 0} \frac{x^2 + x\cdot 0}{x^2 + 0^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$

Since different approaches give different limits, $\boxed{\text{DNE}}$.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$

$x = 0$: $\displaystyle\lim_{y\to 0} \frac{0^2 y}{0^4 + y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$

$y = x^2$: $\displaystyle\lim_{x\to 0} \frac{x^2(x^2)}{x^4 + (x^2)^2} = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$

$\boxed{\text{DNE}}$

3. **[4 points]** Let $f(x, y)$ be a differentiable function. Express $f_y(2, 3)$ as the limit of a difference quotient.

$$f_y(2,3) = \lim_{h \to 0} \frac{f(2,3+h) - f(2,3)}{h}$$

4. **[12 points]** Let $f(x, y) = xe^{x^2 y}$. Find f_x ~~and~~ f_y, and f_{yx}

$$f_x = e^{x^2y} + xe^{x^2y} \cdot y2x = \boxed{e^{x^2y}(1+2x^2y)}$$

$$f_y = xe^{x^2y} \cdot x^2 = \boxed{x^3 e^{x^2y}}$$

$$f_{yx} = 3x^2 e^{x^2y} + x^3 e^{x^2y} \cdot y2x = \boxed{x^2 e^{x^2y}(3+2x^2y)}$$

5. **[12 points]** Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ ~~in~~ where $z^2xy + x\cos(z) = 0$. $F(x,y,z) = z^2xy + x\cos(z)$

$F_x = z^2y + \cos(z)$

$F_y = z^2x$

$F_z = 2zxy - x\sin(z)$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(z^2y + \cos(z))}{2zxy - x\sin(z)} = \boxed{\frac{z^2y + \cos(z)}{x\sin(z) - 2xyz}}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = -\frac{z^2x}{2xyz - x\sin(z)} = \boxed{\frac{z^2x}{x\sin(z) - 2xyz}}$$

6. **[12 points]** Find the equation of the tangent plane to the surface of $z = \cancel{\cos}\,\sin(x^2 - y^2)$ at the point $(1, +1, 0)$.

$$f_x = +\cancel{\sin}\cos(x^2-y^2)\cdot 2x = +2x\,\cancel{\sin}\cos(x^2-y^2)\quad;\quad f_x(1,+1) = (+2)\cdot\cos(0) = 2$$

$$f_y = +\cancel{\sin}\cos(x^2-y^2)\cdot(-2y) = -2y\,\cancel{\sin}\cos(x^2-y^2)\quad;\quad f_y(1,+1) = (-2)(+1)\cos(0) = -2$$

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$z - 0 = 2(x-1) - 2(y-1)$$

$$z = 2x - 2 - 2y + 2\quad;\quad \boxed{z = 2x - 2y}$$

7. **[12 points]** The width w and height h of a rectangle are changing. At a particular instant in time, $w = 5$ cm and is increasing by 2 ~~units per second~~ cm/s, and $h = 8$ ~~and~~ cm is decreasing ~~at~~ by 3 ~~units per second~~ cm/s. Find the rate of change in (i) the area of the rectangle, and (ii) the length of its diagonal.

(i) $A = wh$.

$$\frac{dA}{dt} = \frac{\partial A}{\partial w}\cdot\frac{dw}{dt} + \frac{\partial A}{\partial h}\cdot\frac{dh}{dt} = h\cdot\frac{dw}{dt} + w\cdot\frac{dh}{dt} = 8(2) + (5)(-3)$$

$$= 16 - 15 = 1$$

So, the $\boxed{\text{area is increasing by } 1\ \text{cm}^2/\text{s}.}$

(ii) $L = \sqrt{w^2+h^2}$ $\quad L_w = \frac{1}{2}(w^2+h^2)^{-\frac{1}{2}}\cdot 2w = \frac{w}{\sqrt{w^2+h^2}} = \frac{5}{\sqrt{25+64}} = \frac{5}{\sqrt{89}}$

$$L_h = \frac{1}{2}(w^2+h^2)^{-\frac{1}{2}}\cdot 2h = \frac{h}{\sqrt{w^2+h^2}} = \frac{8}{\sqrt{25+64}} = \frac{8}{\sqrt{89}}$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial w}\cdot\frac{dw}{dt} + \frac{\partial L}{\partial h}\cdot\frac{dh}{dt} = \frac{5}{\sqrt{89}}\cdot(2) + \frac{8}{\sqrt{89}}\cdot(-3) = \frac{10}{\sqrt{89}} - \frac{24}{\sqrt{89}} = \boxed{\frac{-14}{\sqrt{89}}\ \text{cm/s}}$$

8. **[12 points]** Find the directional derivative of $f(x,y) = y^2\tan(x)$ at $(\pi/4, -2)$ in the direction of $2\vec{i} + \vec{j}$.

$\nabla f = \langle f_x, f_y \rangle = \langle y^2 \sec^2(x), 2y \tan(x) \rangle$

$= \langle (-2)^2 \left[\frac{1}{\cos(\pi/4)}\right]^2, 2(-2)\tan(\pi/4) \rangle$

$= \langle 4 \cdot \left[\frac{2}{\sqrt{2}}\right]^2, -4 \cdot 1 \rangle = \langle 8, -4 \rangle$

$D_u f(\pi/4, -2) = \nabla f \cdot \frac{\langle 2, 1\rangle}{\sqrt{2^2+1^2}} = \frac{1}{\sqrt{5}} \langle 8, -4 \rangle \cdot \langle 2, 1 \rangle = \frac{1}{\sqrt{5}}(16-4) = \boxed{\frac{12}{\sqrt{5}}}$

little more space

9. **[12 points]** Find and classify the critical points of $f(x,y) = x^3y + 12x^2 - 8y$ as local minimums, local maximums, or saddle points.

$f_x = 3x^2y + 24x \qquad f_y = x^3 - 8$

$f_y = 0$: $x^3 - 8 = 0$, $x^3 = 8$, $x = 2$

$f_x = 0$: $3x^2y + 24x = 0$
$3 \cdot 4y + 24 \cdot 2 = 0$
$12y + 12 \cdot 4 = 0$
$y + 4 = 0$
$y = -4$.

So, the critical point at $(2, -4)$.

$f_{xx} = 6xy + 24 \qquad f_{xy} = 3x^2 \qquad f_{yy} = 0$

$D = (f_{xx})(f_{yy}) - [f_{xy}]^2 = f_{xx} \cdot 0 - [3x^2]^2 = -9x^4$

$D(2,-4) = -9 \cdot 2^4 < 0$, So $(2,-4)$ is a $\boxed{\text{saddle point}}$.