Math251 **Test 2** **October 7, 2013**

Name: Solutions

Directions: Show all work. No credit for answers without work. Each problem is worth 10 points. Complete 7 of the 8 problems. Your lowest scoring problem is dropped.

1. **[2 parts, 5 points each]** For each part below, determine whether the given three points lie on a straight line.

   (a) $A(2,4,2)$, $B(3,7,-2)$, $C(1,3,3)$

   (b) $D(0,-5,5)$, $E(1,-2,4)$, $F(3,4,2)$

(a) $\overrightarrow{AB} = \langle 1, 3, -4 \rangle$, $\overrightarrow{BC} = \langle -2, -4, 5 \rangle$

Since $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are not scalar multiples of each other, A, B, and C are $\boxed{\text{not on a straight line}}$

(b). $\overrightarrow{DE} = \langle 1, 3, -1 \rangle$, $\overrightarrow{EF} = \langle 2, 6, -2 \rangle$

Since $\overrightarrow{EF} = 2\overrightarrow{DE}$, $\boxed{\text{D, E, and F are on a straight line}}$

2. **[10 points]** Find a unit vector with the same direction as $7\vec{i} - \vec{j} + 3\vec{k}$.

$$|\langle 7, -1, 3 \rangle| = \sqrt{7^2 + (-1)^2 + (3)^2}$$

$$= \sqrt{49 + 1 + 9} = \sqrt{59}$$

So, $\frac{1}{\sqrt{59}}(7\vec{i} - \vec{j} + 3\vec{k}) = \boxed{\frac{7}{\sqrt{59}}\vec{i} - \frac{1}{\sqrt{59}}\vec{j} + \frac{3}{\sqrt{59}}\vec{k}}$

3. **[10 points]** Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where $\vec{a} = \vec{i} - \vec{k}$ and $\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$.

$\vec{a} = \langle 1, 0, -1 \rangle$

$\vec{b} = \langle 3, -2, 1 \rangle$

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(1)(3) + (0)(-2) + (-1)(1)}{\sqrt{(1)^2+(-1)^2} \cdot \sqrt{(3)^2+(-2)^2+(1)^2}}$$

$$= \frac{2}{\sqrt{2 \cdot 14}} = \frac{1}{\sqrt{7}}$$

So $\theta = \cos^{-1}\left(\frac{1}{\sqrt{7}}\right) = \boxed{1.18 \text{ rad}} = \boxed{67.79^\circ}$

4. **[10 points]** Find two unit vectors orthogonal to both $\langle 2, 4, -1 \rangle$ and $\langle 3, -4, 6 \rangle$.

$$\begin{vmatrix} i & j & k \\ 2 & 4 & -1 \\ 3 & -4 & 6 \end{vmatrix} = (24 - (-1)(-4))\vec{i} - (12 - (-1)(3))\vec{j} + (-8 - 12)\vec{k}$$

$$= 20\vec{i} - 15\vec{j} - 20\vec{k}.$$

So $\langle 20, -15, -20 \rangle$ is orthogonal, mult. by $\frac{1}{5}$, we have that $\langle 4, -3, -4 \rangle$ is also orthogonal to both. To get unit vectors:

$$|\langle 4, -3, -4 \rangle| = \sqrt{(4)^2 + (-3)^2 + (-4)^2} = \sqrt{16 + 9 + 16} = \sqrt{41}$$

So $\boxed{\frac{1}{\sqrt{41}}\langle 4, -3, -4 \rangle \text{ and } -\frac{1}{\sqrt{41}}\langle 4, -3, -4 \rangle}$.

5. **[10 points]** Find the equation of the plane that passes through the point $(2, 1, -1)$ and contains the line given by $x = 3 + t$, $y = -1 + 2t$, and $z = 2 + 5t$.

• Find 3 points in plane: $(2,1,-1)$ is given. $\underline{t=0}$: $(3,-1,2)$.

$\underline{t=1}$: $(4,1,7)$.

• Find 2 vectors in plane:

$$\vec{a} = \langle 3,-1,2\rangle - \langle 2,1,-1\rangle = \langle 1,-2,3\rangle$$

$$\vec{b} = \langle 4,1,7\rangle - \langle 2,1,-1\rangle = \langle 2,0,8\rangle$$

• Find normal vector:

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 1 & -2 & 3 \\ 2 & 0 & 8 \end{vmatrix} = (-16-0)\vec{i} - (8-6)\vec{j} + (0-(-4))\vec{k}$$

$$= \langle -16, -2, 4\rangle$$

~~6. **[10 points]** Find the derivative of the vector function given by $\vec{r}(t) = e^{t^2}\vec{i} - \vec{j} + \sin(5t+2)\vec{k}$.~~

• $\underline{\text{Find eqn}}$: $\vec{n} = \langle a,b,c\rangle$ (x_0, y_0, z_0) on plane

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\boxed{\begin{aligned} -16(x-2) - 2(y-1) + 4(z-(-1)) &= 0 \\ -8(x-2) - (y-1) + 2(z+1) &= 0 \\ -8x - y + 2z &= -19 \\ 8x + y - 2z &= 19 \end{aligned}}$$

6.

$$\vec{r}\,'(t) = 2te^{t^2}\vec{i} - 0\vec{j} + 5\cos(5t+2)\vec{k}$$

$$= \boxed{2te^{t^2}\vec{i} + 5\cos(5t+2)\vec{k}}$$

7. **[10 points]** Find the curvature of the curve given by $\vec{r}(t) = 2\sin t\vec{i} - 6t\vec{j} + 2\cos t\vec{k}$.

$$\vec{r}\,' = 2\cos(t)\vec{i} - 6\vec{j} - 2\sin t\vec{k} \;; \qquad |\vec{r}\,'| = \sqrt{4\cos^2 t + 36 + 4\sin^2 t} = \sqrt{40} = 2\sqrt{10}$$

$$\vec{r}\,'' = -2\sin(t)\vec{i} + 0\vec{j} - 2\cos t\vec{k}$$

$$\cancel{K} = \cancel{4}\; \vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} i & j & k \\ 2\cos t & -6 & -2\sin t \\ -2\sin t & 0 & -2\cos t \end{vmatrix} = (12\cos t - 0)\vec{i} - (-4\cos^2 t - 4\sin^2 t)\vec{j} + (0 - 12\sin t)\vec{k}$$

$$= 12\cos t\vec{i} + 4\vec{j} - 12\sin t\vec{k}$$

$$|\vec{r}\,' \times \vec{r}\,''| = \sqrt{144\cos^2 t + 16 + 144\sin^2 t} = \sqrt{160} = 4\sqrt{10}.$$

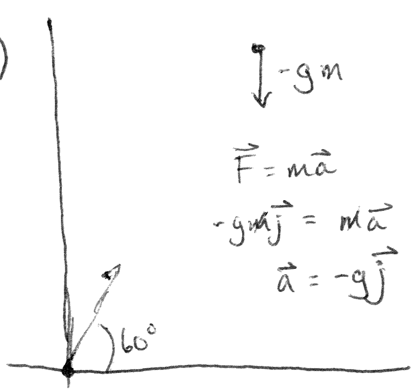
8. At time $t = 0$, a projectile is fired from the origin $(0, 0)$ with an initial speed of 200 m/s at an angle of elevation of $60°$. Recall accel. of gravity $g = 9.8\ m/s^2$

(a) **[7 points]** Find the position function $\vec{r}(t)$ that governs the motion of the projectile.

(b) **[3 points]** ~~Use the position function to~~ Find the maximum height the projectile reaches.

$$\text{So } K = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{4\sqrt{10}}{(2\sqrt{10})^3} = \frac{4\sqrt{10}}{8 \cdot 10 \cdot \sqrt{10}} = \boxed{\frac{1}{20}}$$

8. (a)



$-gm$

$\vec{F} = m\vec{a}$

$-gm\vec{j} = m\vec{a}$

$\vec{a} = -g\vec{j}$

- $\vec{r}_0 = \langle 0, 0 \rangle$ (Initial position)

- $\vec{v}_0 = \langle 200\cos 60°, 200\sin 60° \rangle$
$= \langle 200 \cdot \frac{1}{2}, 200 \cdot \frac{\sqrt{3}}{2} \rangle = \langle 100, 100\sqrt{3} \rangle$

- $\vec{v}(t) = v_0 + \int_0^t \vec{a}\,du = v_0 + \int_0^t -g\vec{j}\,du$

$= \cancel{\langle} 100\vec{i} + 100\sqrt{3}\vec{j} + (-gu)\Big|_0^t \vec{j}$

$= 100\vec{i} + 100\sqrt{3}\vec{j} - gt\vec{j}$

$= 100\vec{i} + (100\sqrt{3} - gt)\vec{j}$

8 cont.

$$\vec{r}(t) = \vec{r}_0 + \int_0^t \vec{v}(u)\, du$$

$$= 0\vec{i} + 0\vec{j} + \int_0^t 100\vec{i} + (100\sqrt{3} - gu)\vec{j}\, du$$

$$= \left(100u\Big|_0^t\right)\vec{i} + \left(100\sqrt{3}\,u - \tfrac{g}{2}u^2\right)\Big|_0^t \vec{j}$$

$$= \boxed{100t\,\vec{i} + \left(100\sqrt{3}\,t - \tfrac{g}{2}t^2\right)\vec{j}}$$

$$= \boxed{100t\,\vec{i} + (100\sqrt{3}\,t - 4.9t^2)\vec{j}}$$

(b) Projectile at maximum height when the $\vec{j}$ component of velocity is 0:

$100\sqrt{3} - gt = 0$; $\quad gt = 100\sqrt{3}$, $\quad t = \frac{100\sqrt{3}}{g}$ = time of max. height

So, max height = $\vec{j}$ component of $\vec{r}(t)$ at $t = \frac{100\sqrt{3}}{g}$

$$= 100\sqrt{3} \cdot \frac{100\sqrt{3}}{g} - \frac{g}{2} \cdot \left(\frac{100\sqrt{3}}{g}\right)^2$$

$$= \frac{(10{,}000)(3)}{g} - \frac{(10{,}000)(3)}{2g} = \frac{30{,}000}{2g} = \frac{15{,}000}{g} \approx \boxed{1530.6\text{ m}}$$