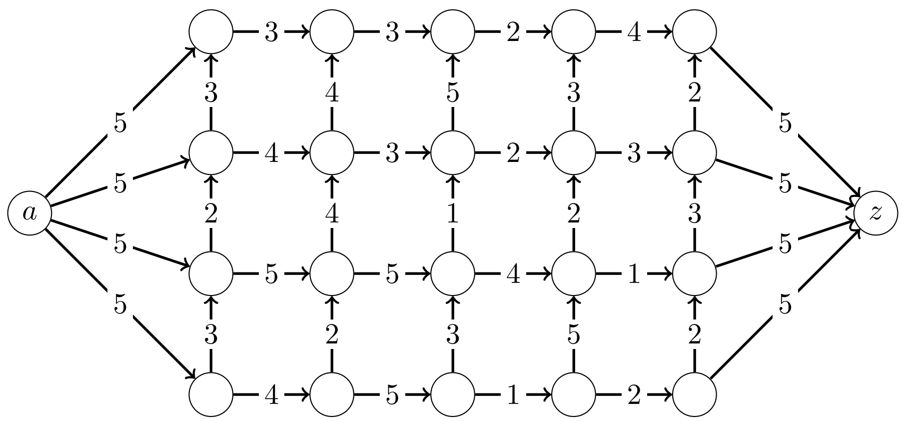
Name: ______________________________

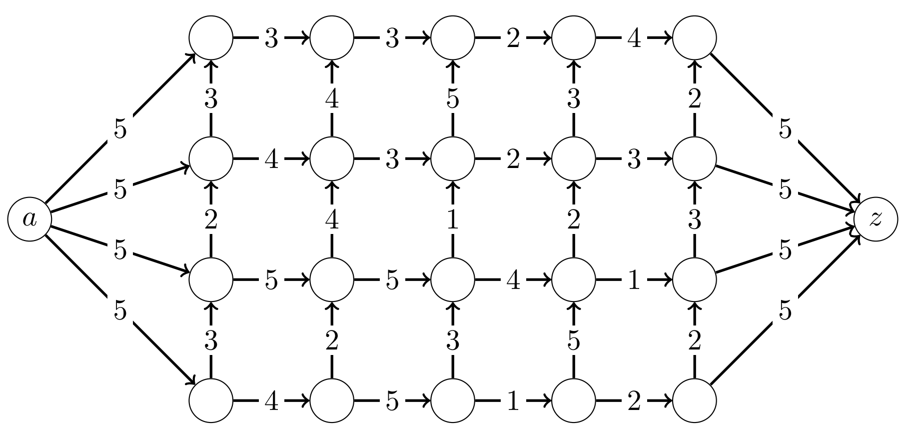
**Directions:** Show all work. Answers without work generally do not earn points. This test has 60 points but is scored out of 50 (higher scores capped at 50).

1. **[2 parts, 5 points each]** Max flow/Min Cut.

   (a) Find a flow of maximum value in the following network.

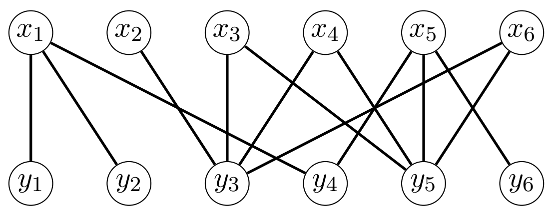



   (b) Find a cut of minimum capacity in the same network (repeated below).



2. **[4 points]** A supermarket stocks 5 different vegetables (broccoli, carrots, lettuce, onions, and peas). At the same time, 5 customers (Beth, Eric, Jerry, Kim, and Sam) enter the store in search of vegetables. Unfortunately, due to limited supplies, it is not possible for more than 1 person to purchase any given vegetable. Sam and Jerry each like 2 vegetables. Kim and Beth each like 3 vegetables. Eric likes all 5 vegetables. Is it guaranteed that everyone can purchase a vegetable they like? Either show that this is the case or find a counterexample.

3. **[3 parts, 2 points each]** Let $G$ be the following bipartite graph.




(a) Find $R(\{x_1, x_3, x_4\})$.

(b) What is the deficiency of $\{x_2, x_3, x_4, x_6\}$?

(c) What can you conclude from part (b) about the size of a maximum matching in $G$?

4. Stable Matchings.

(a) [**7 points**] Given a set $\{1, 2, 3, 4, 5\}$ of men and a set $\{a, b, c, d, e\}$ of women with the following preference lists, find a stable matching.

| | | | |
|---|---|---|---|
| $cbdae$ | 1 | $a$ | 23145 |
| $bcdae$ | 2 | $b$ | 41325 |
| $dcbea$ | 3 | $c$ | 25413 |
| $bcdae$ | 4 | $d$ | 23145 |
| $cbaed$ | 5 | $e$ | 41253 |

(b) [**3 points**] Which (if any) of the matched pairs are common to all stable matchings? (The preference lists are repeated below.)

| | | | |
|---|---|---|---|
| $cbdae$ | 1 | $a$ | 23145 |
| $bcdae$ | 2 | $b$ | 41325 |
| $dcbea$ | 3 | $c$ | 25413 |
| $bcdae$ | 4 | $d$ | 23145 |
| $cbaed$ | 5 | $e$ | 41253 |

5. **[4 points]** Is it possible for a stable matching to pair two people who are each others least desirable partners? Either argue that this is impossible or provide an example where this occurs.

6. **[4 parts, 1.5 points each]** Which of the following are groups? Answer yes or no for each; no justification required. Here, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ is the set of positive real numbers, $+$ denotes standard arithmetic addition, and $\times$ denotes standard arithmetic multiplication.

   (a) $(\mathbb{R}, +)$

   (b) $(\mathbb{R}^+, +)$

   (c) $(\mathbb{R}, \times)$

   (d) $(\mathbb{R}^+, \times)$

7. **[2 points]** If the received transmission is $r = 00110110$ and the error pattern is $e = 00110011$, what transmission was sent?

8. Let $W = \mathbb{Z}_2^5$, so our messages are bitstrings of length 5. Consider the parity bit encoding scheme $E\colon W \to \mathbb{Z}_2^6$ given by $E(x_1 \cdots x_5) = x_1 \cdots x_5 y$ where $y = x_1 + \cdots + x_5 \mod 2$. The decoding function $D(x_1 \cdots x_5 y)$ first checks whether the parity bit $y$ is correct; if it is, then the decoding function returns $x_1 \cdots x_5$ as the transmitted message. Otherwise, the decoding function reports an error. The transmission channel flips bits with probability $p = 0.1$.

   (a) **[2 points]** If a message $x \in W$ is sent *without* any encoding, what is the probability that it is received and decoded properly?

   (b) **[2 points]** A message is sent using the encoding scheme. If the received message is 011011, what does the decoding function do?

   (c) **[4 points]** A message is sent using the encoding scheme. There are three possibilities: either the message is correctly decoded, there is an *undetected* error in transmission, or there is a detected error in transmission. Find the probabilities of each of these 3 cases.

9. **[2 points]** List the elements in $S(1001, 1)$.

10. **[2 points]** Let $x \in \mathbb{Z}_2^{12}$. How many elements are in $S(x, 4)$? You may leave your answer in terms of binomial coefficients.

11. Consider the encoding function $E\colon \mathbb{Z}_2^2 \to \mathbb{Z}_2^6$ given by

$$E(00) = 000000 \qquad E(01) = 001111$$
$$E(10) = 111100 \qquad E(11) = 110011$$

(a) **[2 points]** If our goal is to detect errors, how many errors can we tolerate?

(b) **[1 point]** If our goal is to detect errors and 110111 is received, what should the decoding function do?

(c) **[2 points]** If our goal is to correct errors, how many errors can we tolerate?

(d) **[1 point]** If our goal is to correct errors and 101111 is received, what should the decoding function do?