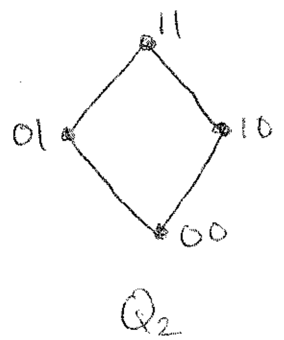
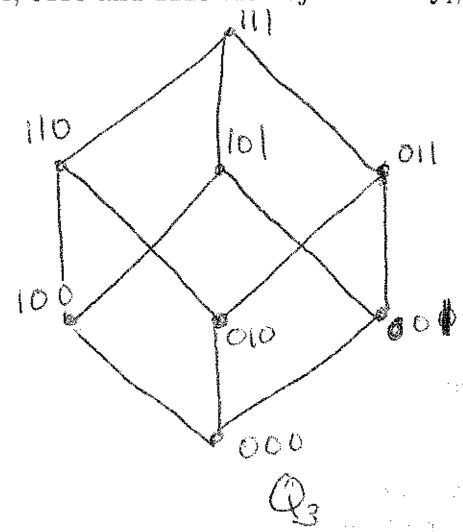
Name: Solutions

Directions: Show all work. Answers without work generally do not earn points. This test has 60 points but is scored out of 50 (higher scores capped at 50).

1. **[3 parts, 4 points each]** Recall the n-dimensional hypercube Q_n is the graph whose vertices are the set of all bitstrings of length n where $x_1 \ldots x_n$ and $y_1 \ldots y_n$ are adjacent if and only if they differ in exactly one coordinate. (For example, 0110 and 1110 are adjacent in Q_4, but 0110 and 1111 are not.)

   (a) Draw Q_2 and Q_3.





   (b) Give a formulas for the number of vertices and the number of edges in Q_n.

$$\# \text{ vertices} = \boxed{2^n}$$

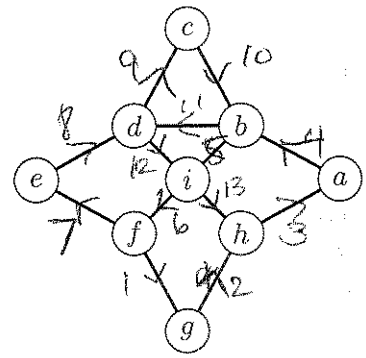
$$2|E(G)| = \sum_{v} d(v) = \sum_{v} n = n \cdot 2^n$$

$$|E(G)| = \frac{n 2^n}{2} = \boxed{n 2^{n-1}}$$

   (c) For which n does Q_n have an Eulerian circuit? Explain.

A graph ~~has~~ an Eulerian circuit if and only if all edges are in the same component (true for all Q_n) and ~~of~~ all vertices have even degree (true for Q_n when n is even.) Therefore Q_n has an Eulerian circuit if and only if $\boxed{n \text{ is even}}$.

2. **[4 points]** Find an Eulerian trail in the following graph.

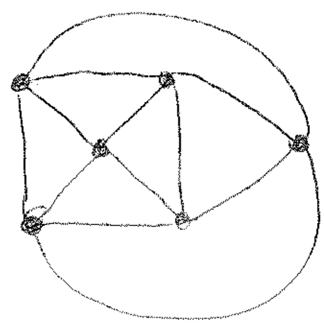


f and h have odd degree, so these are endpoints of the trail.

$\boxed{f\ g\ h\ a\ b\ i\ f\ e\ d\ c\ b\ d\ i\ h}$

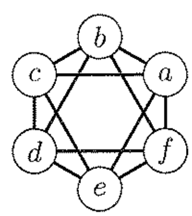
Many other answers possible.

3. **[4 points]** Give an example of a 4-regular planar graph without loops or multiple edges. (Recall that a graph is k-regular if every vertex has degree k.)

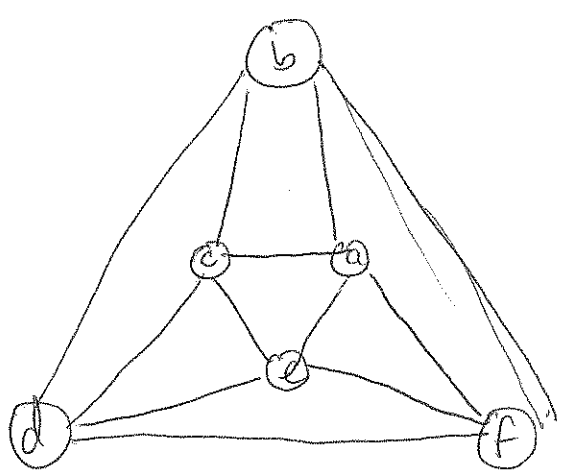


(The octahedron)

4. **[4 points]** Is the following graph planar or nonplanar? If it is planar, give a planar drawing. If not, find a subgraph homeomorphic to K_5 or $K_{3,3}$.



Planar:



Also the octahedron!

5. **[2 parts, 4 points each]** Let G be a 10-vertex graph with 26 edges and without loops or multiple edges.

   (a) Show that G is not a planar graph.

   G is planar $\Longrightarrow |E(G)| \leq 3(10)-6 = 24$.

   Since $|E(G)| = 26 > 24$, G is not planar.

   (b) Show that in every drawing of G in the plane, at least *three* edges of G are involved in edge crossings.

   ~~Lets~~ Consider a drawing of G in the plane. Because G is not planar, there is an edge crossing involving two edges e_1 and e_2. Let $G' = G - e_1$. Note that G' is also not planar, because $|E(G')| = 25 > 24$. So two edges of G', say f_1 and f_2, are ~~in an~~ involved in a crossing. Therefore in G, $\{e_1, f_1, f_2\}$ are all involved in edge crossing

6. **[4 points]** Let G be a connected planar graph with 328 edges without loops and multiple edges. In a planar drawing of G, the boundary of every region contains at least 8 edges. How many vertices must G contain? Give the best lower bound you can.

Length Sum:

$$\sum_{F \text{ is a face}} \text{length}(F) = 2|E(G)|$$

$$8f \leq 2e, \qquad f \leq \frac{e}{4} = \frac{328}{4} = 82$$

$n - e + f = 2$

$n = 2 + e - f$

$n = 2 + 328 - f$

$n \geq 330 - 82$

$\boxed{n \geq 248}$

Note: this is sharp; there are such graphs with 248 vertices.

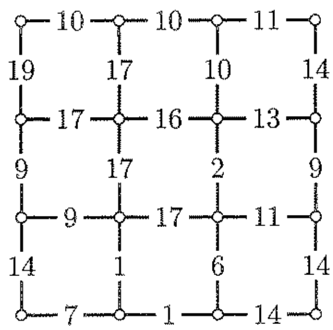
7. **[4 points]** A connected 50-vertex graph G with 100 edges is drawn in the plane. How many regions are there in the drawing?

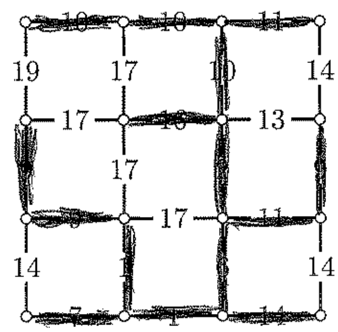
$n - e + f = 2$

$50 - 100 + f = 2$

$\boxed{f = 52}$

8. **[4 points]** Find a minimum weight spanning tree in the following graph (2 copies).

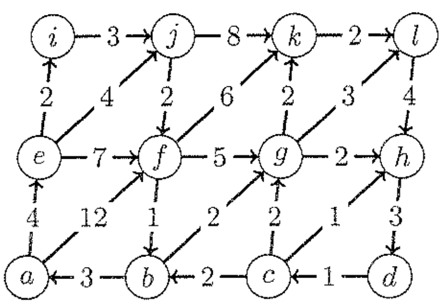


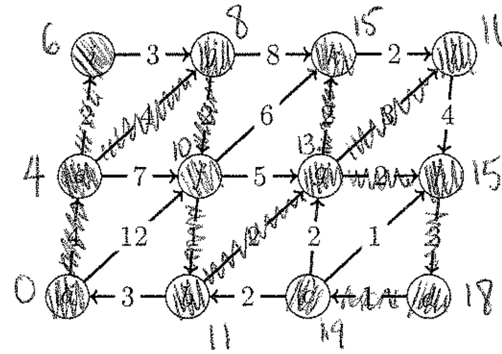


9. **[4 points]** Let G be a connected graph without loops or multiple edges on at least 3 vertices. Suppose that G has distinct edge weights, and let e be the *second* lightest edge. Prove or disprove: the minimum weight spanning tree of G contains e.

This is true. In Kruskal's algorithm, we first select the lightest edge. Next, Kruskal considers e. Because e cannot make a cycle with the only other edge already selected, Kruskal's Alg selects e. Therefore e is in the minimum weight spanning tree.

10. **[2 parts, 3 points each]** Consider the following directed graph (2 copies).





(a) Use Dijkstra's algorithm to find the distance between a and all other vertices.

a	b	c	d	e	f	g	h	i	j	k	l
0	11	19	18	4	10	13	15	6	8	15	16

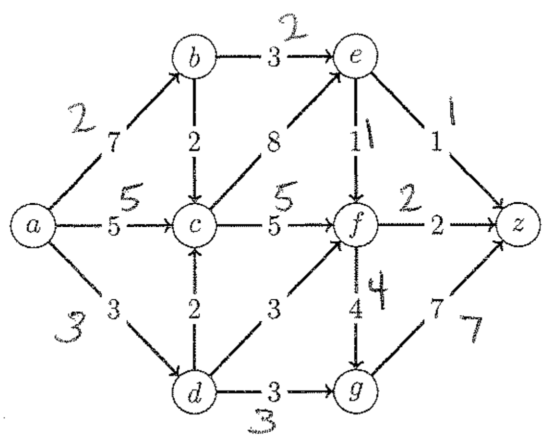
(b) Find a shortest path from a to k.

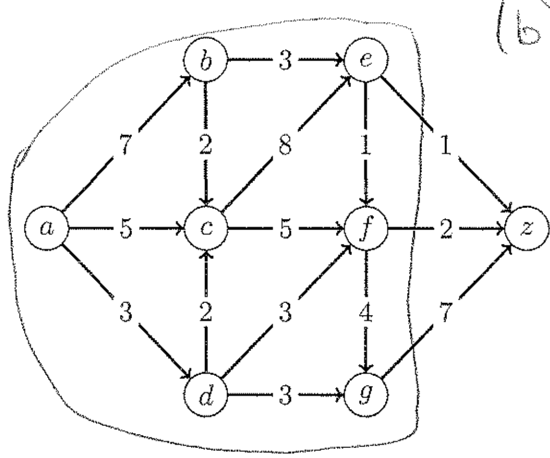
a e j f b g k

11. **[2 parts, 3 points each]** Consider the following network N (2 copies).

a) Flow values.

Other answers possible





(b) Cut with capacity 10.

Other answer possible.

(a) Find a flow in N with value 10. (Indicate flow values on a copy of the network above; clearly mark which copy contains your answer.)

(b) Find a cut $(P, \overline{P})$ in N of capacity 10. (Indicate the cut by circling a set of vertices in the network above; clearly mark which circle represents the cut.)