Directions: Homework 4 consists of these exercises and others as assigned in the text.

1. In class on Sept. 7, we showed that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k-1}$ by counting the k-element subsets of $\{1, \ldots, n\}$ in two different ways. The LHS counts this directly. For the RHS, we used the rule of sum to split the k-element subsets of $\{1, \ldots, n\}$ into a red group and a blue group. The red group consisted of all k-element subsets of $\{1, \ldots, n\}$ that contain n; there are $\binom{n-1}{k-1}$ of these. The blue group consisted of all k-element subsets of $\{1, \ldots, n\}$ that do not contain n; there are $\binom{n-1}{k}$ of these.

   This problem will illustrate how the counting argument works when $n = 5$ and $k = 3$.

   (a) List all 3-element subsets of $\{1, 2, 3, 4, 5\}$. How many are in your list? Compare this with $\binom{n}{k}$, which equals $\binom{5}{3}$.

   (b) List all of the red subsets; i.e., the 3-element subsets that contain n. How many are in your list? Compare this with $\binom{n-1}{k-1}$, which equals $\binom{4}{2}$.

   (c) List all of the blue subsets; i.e., the 3-element subsets that do not contain n. How many are in your list? Compare this with $\binom{n-1}{k}$, which equals $\binom{4}{3}$.

2. Give a combinatorial argument that $\binom{n}{k} = \binom{n}{n-k}$.

   Hint 1: Describe a way to pair each k-element subset of $\{1, \ldots, n\}$ with an $(n-k)$-element subset of $\{1, \ldots, n\}$. Make sure that each subset belongs to exactly one pair.

   Hint 2: If you are having trouble with this problem, try it with $n = 5$ and $k = 3$. Write down all 3-element subsets of $\{1, 2, 3, 4, 5\}$ and all 2-element subsets of $\{1, 2, 3, 4, 5\}$. Next, try to pair the 3-element subsets and the 2-element subsets in a natural way. Then, describe how the argument works in the general case.

3. A class with n boys and n girls needs to form a focus group consisting of n students. By counting the number of possible focus groups in two ways, prove that

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \binom{2n}{n}.$$