Name: ____________________________

Directions: Show all work. Answers without work generally do not earn points. This test has 60 points, but is scored out 50 (scores capped at 50).

1. [**6 points**] Recall that M_{22} is the vector space of (2×2)-matrices. Either show that the following matrices are linearly independent or express the zero vector as a non-trivial linear combination.
$$\begin{bmatrix} 5 & 3 \\ 8 & -1 \end{bmatrix}, \begin{bmatrix} 11 & 5 \\ 12 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}$$

2. [**4 parts, 1.5 points each**] Let S be a subset of M_{22} of size n, so that $S = \{v_1, \ldots, v_n\}$ where each v_j is a (2×2)-matrix. In (a)–(d) below, you do not need to show any work.
   (a) If S is linearly independent, what (if anything) can you say about n?
   (b) If S is linearly dependent, what (if anything) can you say about n?
   (c) If S spans M_{22}, what (if anything) can you say about n?
   (d) If S does not span M_{22}, what (if anything) can you say about n?

3. **[2 parts, 3 points each]** Consider P_3, the vector space of polynomials of degree at most 3.

   (a) Find a basis for P_3. What is the dimension of P_3?

   (b) Let V be the subspace of P_3 consisting of all polynomials $p(t)$ such that $p(t) = p(-t)$ for every real number t. Find a basis for V. (*Hint:* consider a general element $p(t) = at^3 + bt^2 + ct + d$. What must be true for $p(t) = p(-t)$ to hold?)

4. **[6 points]** A matrix A and its reduced row-echelon form are displayed below.

$$A = \begin{bmatrix} 1 & 1 & 7 & 1 & 1 & 1 \\ -1 & 1 & 3 & 3 & 0 & -3 \\ 2 & 1 & 9 & 0 & 1 & 4 \\ -2 & 1 & 1 & 4 & 1 & -8 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & -1 & 0 & 3 \\ 0 & 1 & 5 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   Find bases for the row space, the column space, and the null space of A. Clearly label which basis is for which space.

5. **[6 points]** Consider the linear transformation $f\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ given by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Interpret the transformation f graphically, as a function mapping points in the plane to other points in the plane.

6. **[6 points]** Let $L\colon\ \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad L\left(\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

If possible, compute $L\left(\begin{bmatrix} 3 \\ 9 \\ 5 \end{bmatrix}\right)$.

7. **[2 parts, 6 points each]** Let $L\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + u_2 \\ u_2 + u_3 \end{bmatrix}.$$

(a) Find a basis for $\ker L$.

(b) Let S and T be ordered bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ given by

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Find the matrix A that represents L with respect to S and T.

8. **[2 parts, 6 points each]** Let $L\colon V \to W$ be a linear transformation, let $S = \{v_1, \ldots, v_n\}$ where each v_i is a vector in V, and let $T = \{L(v_1), \ldots, L(v_n)\}$. One of the following statements is true and the other is false.

   - If S is linearly independent in V, then T is linearly independent in W.
   - If T is linearly independent in W, then S is linearly independent in V.

   (a) Identify the true statement and prove it.

   (b) Identify the false statement and give a counterexample. Your counterexample should consist of a vector space V, a vector space W, a linear transformation $L\colon V \to W$, and appropriate sets S and T. *Hint:* the simpler, the better.