Name: ______________________________

Directions: Show all work. Answers without work generally do not earn points. This test has 60 points, but is scored out 50 (scores capped at 50).

1. **[3 parts, 3 points each]** Find the determinant of the following matrices.

(a) $\begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 1 & 5 \\ 2 & -1 & 1 \\ 1 & -2 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 & 1 & 4 \\ 5 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 2 & 3 & 3 & 1 \end{bmatrix}$

2. **[5 points]** If $A^2 = A$, what are the possible values for $\det(A)$? *Hint:* what is $\det(A^2)$ in terms of $\det(A)$?

3. **[5 points]** Find a real number a such that $\begin{bmatrix} 5 & 1 & 4 \\ 2 & -2 & 3 \\ -3 & 1 & a \end{bmatrix}$ is singular. *Hint:* what is the connection between singular matrices and determinants?

4. **[5 points]** Find values a and b such that $\begin{bmatrix} a-2b \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2a+b \\ b-a \end{bmatrix}$ are the same vector, and sketch this vector in $\mathbb{R}^2$. Use the horizontal axis for the first/top coordinate and the vertical axis for the second/bottom coordinate.

5. **[5 points]** Let V be the set of all real (2×1)-matrices, and define operations $\oplus$ and $\odot$ as follows:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 + y_2 \\ x_1 + y_1 \end{bmatrix} \qquad\qquad r \odot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}.$$

Find one defining property of vector spaces that $(V, \oplus, \odot)$ lacks. Justify your answer.

6. [**5 points**] Let V be a real vector space. Prove that exactly one element $\vec{0}$ in V has the property that $\vec{0} \oplus \mathbf{u} = \mathbf{u} \oplus \vec{0} = \mathbf{u}$ for each $\mathbf{u} \in V$.

7. [**5 points**] Let V be a real vector space. Prove that if $\mathbf{u} \oplus \mathbf{u} = \vec{0}$, then $\mathbf{u} = \vec{0}$.

8. **[5 points]** Let W be the set of all real (2×2)-matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + d = c + b$. Is W a subspace of the vector space M_{22} of all (2×2)-matrices? Justify your answer.

9. Recall that P_2 is the vector space of all polynomials of degree at most 2.

   (a) **[3 points]** Give a small spanning subset of P_2.

   (b) **[5 points]** Let $S = \{-t^2 + 4, 2t + 1, t^2 + t + 1\}$. Is $2t^2 + t$ in span S? Justify your answer.

10. **[5 points]** Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 5 & 6 \\ 1 & -1 & 7 & 9 \end{bmatrix}$. Find vectors that span the null space of A.

11. **[3 points]** Let A be a real $(n \times n)$-matrix with $\det(A) = k$. Find a formula for $\det(A + A)$.