Name: Solutions

Directions: Show all work. Answers without work generally do not earn points. This test has 60 points, but is scored out 50 (scores capped at 50).

1. **[3 parts, 3 points each]** Find the determinant of the following matrices.

(a) $\begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$

$2 \cdot 5 - (2)(-1) = \boxed{12}$

(b) $\begin{bmatrix} 3 & 1 & 5 \\ 2 & -1 & 1 \\ 1 & -2 & -4 \end{bmatrix}$

$3(-1)(-4) + (1)(1)(1) + 5(2)(-2)$

$- \left[(1)(-1)(5) + (-2)(1)(3) + (-4)(2)(1) \right]$

$= 12 + 1 - 20 - [-5 - 6 - 8]$

$= 12 - 19 + 19$

$= \boxed{12}$

(c) $\begin{bmatrix} 2 & 3 & 1 & 4 \\ 5 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 2 & 3 & 3 & 1 \end{bmatrix}$

Row expansion by 3^{rd} row:

$0|\sim| - 0|\sim| + 0|\sim| - 2\begin{vmatrix} 2 & 3 & 1 \\ 5 & 1 & 2 \\ 2 & 3 & 3 \end{vmatrix}$

$= -2\left[(6 + \cancel{12} + 15) - (2 + \cancel{12} + 45)\right]$

$= -2[4 - 30]$

$= -2[-26] = \boxed{52}$

2. **[5 points]** If $A^2 = A$, what are the possible values for $\det(A)$? *Hint:* what is $\det(A^2)$ in terms of $\det(A)$?

$\det(A^2) = \det(A)$

$\det(A) \cdot \det(A) = \det(A)$

$\det(A)[\det(A) - 1] = 0$

$\boxed{\det(A) = 0 \text{ or } \det(A) = 1}$

3. **[5 points]** Find a real number a such that $\begin{bmatrix} 5 & 1 & 4 \\ 2 & -2 & 3 \\ -3 & 1 & a \end{bmatrix}$ is singular. *Hint:* what is the connection between singular matrices and determinants?

Want determinant to be 0:

$5(-2)a + 1(3)(-3) + 4(2)(1) - [(-3)(-2)4 + (1)(3)(5) + a(2)(1)] =$

$-10a - 1 - [24 + 15 + 2a] = 0$

$-12a - 40 = 0$

$-12a = 40$

$a = -\frac{40}{12} = \boxed{-\frac{10}{3}}$

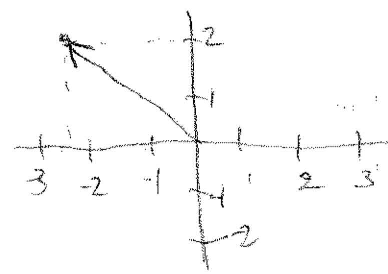
4. **[5 points]** Find values a and b such that $\begin{bmatrix} a-2b \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2a+b \\ b-a \end{bmatrix}$ are the same vector, and sketch this vector in $\mathbb{R}^2$. Use the horizontal axis for the first/top coordinate and the vertical axis for the second/bottom coordinate.

$$a-2b=2a+b$$
$$2=b-a$$

$$0=a+3b$$
$$2=-a+b$$
$$2=4b$$

$$b=\tfrac{1}{2},\quad a=-\tfrac{3}{2}$$

Vector: $\begin{bmatrix} a-2b \\ 2 \end{bmatrix} = \begin{bmatrix} -\tfrac{3}{2}-2(\tfrac{1}{2}) \\ 2 \end{bmatrix} = \begin{bmatrix} -\tfrac{5}{2} \\ 2 \end{bmatrix}$



5. **[5 points]** Let V be the set of all real (2×1)-matrices, and define operations $\oplus$ and $\odot$ as follows:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2+y_2 \\ x_1+y_1 \end{bmatrix} \qquad r\odot\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix}.$$

Find one defining property of vector spaces that $(V,\oplus,\odot)$ lacks. Justify your answer.

Associativity of $\oplus$:

$$\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) \oplus \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \oplus \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right)$$

$$\begin{bmatrix} x_2+y_2 \\ x_1+y_1 \end{bmatrix} \oplus \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} y_2+z_2 \\ y_1+z_1 \end{bmatrix}$$

$$\begin{bmatrix} x_1+y_1+z_2 \\ x_2+y_2+z_1 \end{bmatrix} \overset{?}{\cancel{=}} \begin{bmatrix} x_2+y_1+z_1 \\ x_1+y_2+z_2 \end{bmatrix} \quad \underline{\text{No}}.$$

A Distributive property also fails: $(r+s)\odot\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq r\odot\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus s\odot\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

6. **[5 points]** Let V be a real vector space. Prove that exactly one element $\vec{0}$ in V has the property that $\vec{0} \oplus \mathbf{u} = \mathbf{u} \oplus \vec{0} = \mathbf{u}$ for each $\mathbf{u} \in V$.

Suppose that two vectors v and w have this property.

Then $v \oplus w = v$, by the additive identity properties of w.

Also, $v \oplus w = w$, by the additive identity properties of v.

Therefore $v = w$.

7. **[5 points]** Let V be a real vector space. Prove that if $\mathbf{u} \oplus \mathbf{u} = \vec{0}$, then $\mathbf{u} = \vec{0}$.

$\vec{0} = u \oplus u = (1 \odot u) \oplus (1 \odot u)$ [Prop # 8; mul. ident.

$= (1+1) \odot u$ [Distributive prop]

$= 2 \odot u$ alg.

Now, ~~mu~~ multiply both sides of $2 \odot u = \vec{0}$ by $\frac{1}{2}$:

$\frac{1}{2} \odot (2 \odot u) = \frac{1}{2} \odot \vec{0}$

$(\frac{1}{2} \cdot 2) \odot u = \vec{0}$

$1 \odot u = \vec{0}$

$u = \vec{0}$.

8. **[5 points]** Let W be the set of all real (2×2)-matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + d = c + b$. Is W a subspace of the vector space M_{22} of all (2×2)-matrices? Justify your answer.

Yes: Closure under $\oplus$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}.$$

$a+d = b+c$

$e+h = f+g$

$a+e+d+h \stackrel{?}{=} b+f+c+g$

$(a+d)+(e+h) = (b+c)+(f+g)$ ✓.

Closure under $\odot$:

$$r \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$$

$a+d = c+b$

$ra + rd \stackrel{?}{=} rc + rb$

$r(a+d) = r(c+b)$ ✓

9. Recall that P_2 is the vector space of all polynomials of degree at most 2.

(a) **[3 points]** Give a small spanning subset of P_2.

$$\{1, t, t^2\}$$

(b) **[5 points]** Let $S = \{-t^2+4, 2t+1, t^2+t+1\}$. Is $2t^2+t$ in span S? Justify your answer.

$$a(-t^2+4) + b(2t+1) + c(t^2+t+1) = 2t^2+t$$

$$[-a+c]t^2 + [2b+c]t + [4a+b+c] = 2t^2+t$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Extra Sheet #9(b)

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 4 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 5 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 5 & 8 \\ 0 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & -9 & -15 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & 1 & 5/3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{24}{3} - \frac{25}{3} \\ 0 & 0 & 1 & 5/3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 5/3 \end{array}\right]$$

Yes, $2t^2+t$ is in the span.

$$\left(-\frac{1}{3}\right)\left(-t^2+4\right) + \left(-\frac{1}{3}\right)\left(2t+1\right) + \left(\frac{5}{3}\right)\left(t^2+t+1\right) = 2t^2+t$$

10. **[5 points]** Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 5 & 6 \\ 1 & -1 & 7 & 9 \end{bmatrix}$. Find vectors that span the null space of A.

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 3 & 4 \\ 0 & -2 & 6 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\cancel{x_1 + x_2 + x_3 + x_4 = 0}$

$\cancel{x_2 - 3x_3 - 4x_4 = 0}$

$\cancel{0 = 0}$

$\cancel{x_4 = a,\ x_3 = b,\ x_2 = 3b + 4a}$

$\cancel{x_1 =}$

~~Set null space of A is {~~

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 4 & 5 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 4x_3 + 5x_4 = 0 \qquad x_4 = b,\ x_3 = a$

$x_2 - 3x_3 - 4x_4 = 0 \qquad x_2 = 3a + 4b$

$0 = 0 \qquad x_1 = -4a - 5b$

$$null(A) = \left\{ \begin{bmatrix} -4a - 5b \\ 3a + 4b \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\} = \left\{ a \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -5 \\ 4 \\ 0 \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$= \boxed{\text{span} \left\{ \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

11. **[3 points]** Let A be a real $(n \times n)$-matrix with $\det(A) = k$. Find a formula for $\det(A + A)$.

$$\det(A + A) = \det(2A) = \det\left(\begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & & 0 \\ 0 & 0 & 2 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix} \cdot A \right)$$

$$= \det\left(\begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 2 \end{bmatrix} \right) \cdot \det(A)$$

$$= 2^n \cdot k = \boxed{2^n k}$$