Announcements

- Temp. course webpage: http://www.bigw.org/~kmilans
- HW1, and Quiz 1 will be offline.
- HW 1 assigned on Tues; not collected or graded
- Quiz 1 – in Class – on Monday Aug 30.
- Office hours 3-4pm LeConte 307

Sections 1.2, 1.3: Linear Functions and Average Rate of Change

- A linear function is a function

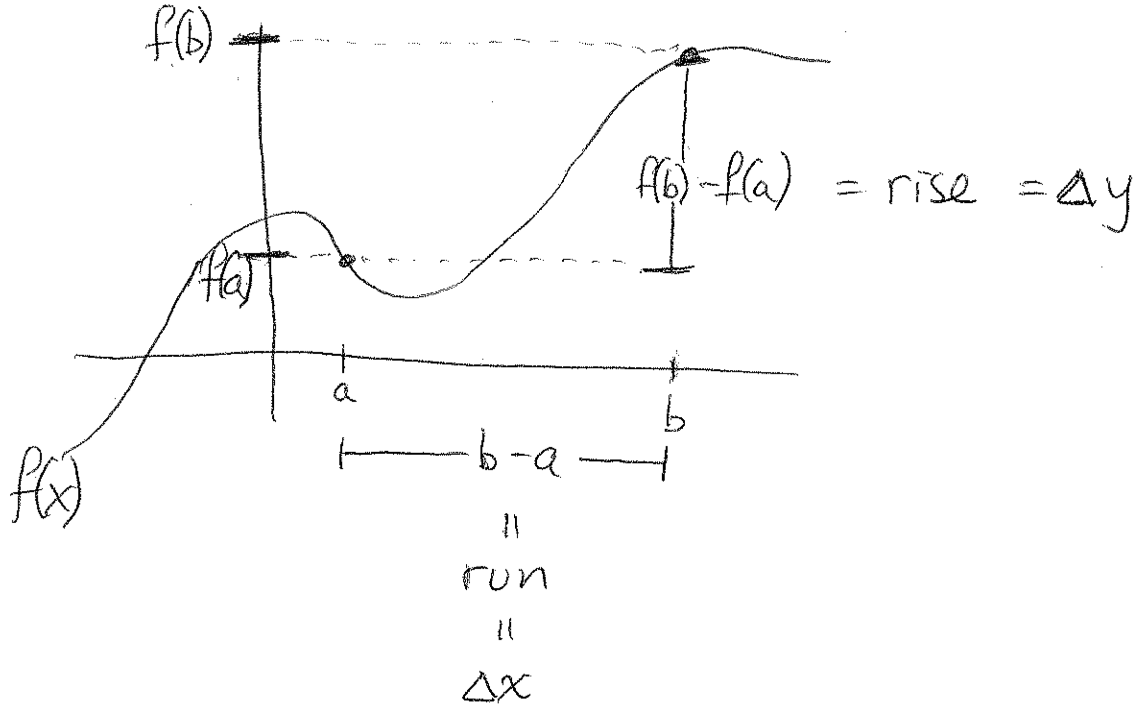
  (i) whose graph is a line.

  (2) that has the form $y = f(x) = mx + b$, where m is the slope, and b is the y-intercept or vertical intercept.

  (3) that has uniform average rate of change.

Average Rate of Change

The average rate of change of a function f between $x = a$ and $x = b$ is given by the difference quotient $\boxed{\dfrac{f(b) - f(a)}{b - a}} = \dfrac{\text{rise}}{\text{run}} = \dfrac{\Delta y}{\Delta x}$



<u>Ex:</u> $f(x) = x^2 + 1$. Find the average rate of f

(1) between $x = 0$ and $x = 2$
(2) between $x = 1$ and $x = 2$.

<u>Soln:</u> <u>(1)</u> $$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(0)}{2-0} = \frac{(2^2+1)-(0^2+1)}{2}$$

$$= \frac{5-1}{2} = \boxed{2}$$

(2) $$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(1)}{2-1}$$

$$= \frac{5-2}{1} = \boxed{3}$$

$\Rightarrow$ f does not have uniform average rate of change

$\Rightarrow$ f is not linear.

Ex: $f(x) = -3x + 7$. ~~Find~~ to find the average rate of change of f between $x = a$ and $x = b$.

Soln: $$\frac{f(b)-f(a)}{b-a} = \frac{(-3b+7)-(-3a+7)}{b-a}$$

$$= \frac{-3b \cancel{+7} + 3a \cancel{-7}}{b-a}$$

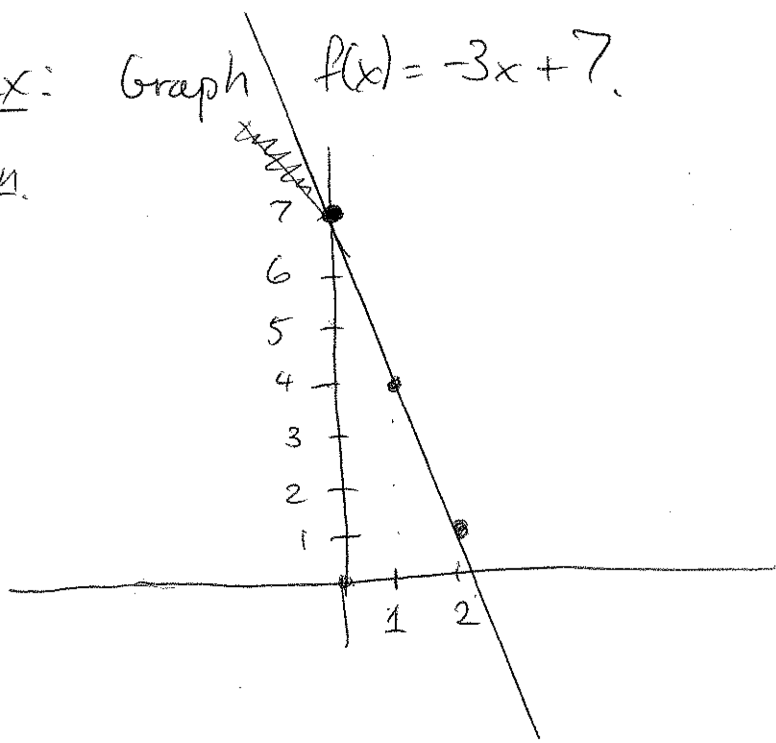
$$= \frac{-3b+3a}{b-a}$$

$$= \frac{-3(b-a)}{b-a} = \frac{-3\cancel{z}}{\cancel{z}}, \text{ where } z = b-a$$

$$= \boxed{-3}$$

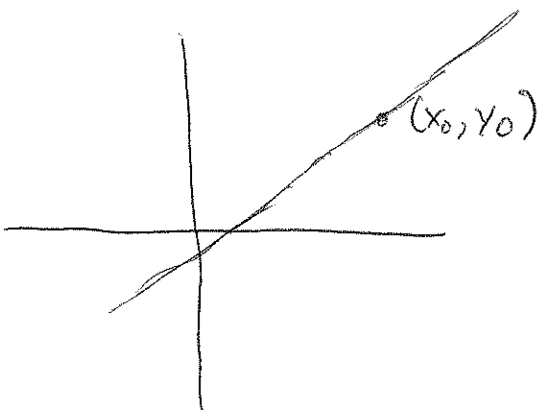
Ex: Graph $f(x) = -3x + 7$.

Soln.



• y-intercept 7.
(i.e. $f(0) = 7$)

Point Slope Form



- Knowing 1 point on the graph of the function and the slope determines a linear function.
- Know slope m
- Know point (x_0, y_0)

Point-Slope Form: $$y - y_0 = m(x - x_0)$$

Example: A car is leased. For each mile driven beyond 36,000 the dealer charges \$0.15. For $x \geq 36{,}000$, let $C(x)$ be the cost (in dollars) ~~of~~ charged for driving x miles.

(1) Give a table with values of $C(x)$ when $x = 36{,}000$, $x = 36{,}100$, and $x = 36{,}200$.

(2) Determine the formula for $C(x)$.

Soln (1)

x	36,000	36,100	36,200
$y = C(x)$	0	15	30

100 (change in x), 15 (change in y)

(2) $m = \dfrac{\Delta y}{\Delta x} = \dfrac{15}{100} = 0.15$ // $(x_0, y_0) = (36{,}000,\ 0)$

$y - 0 = 0.15(x - 36{,}000)$ // $\boxed{y = 0.15x - 5400}$

Announcements

- HW 1 on website; discussed on ~~Monday~~ Friday
- Quiz 1 in class on Monday (Sec 1.1, 1.2, 1.3)
- Today: Finish 1.2 and 1.3

Absolute and Relative Change

- Suppose the price of a cake increases from \$4 to \$5.50.
- Absolute Change = (new value) − (old value)

$$= 5.5 - 4 = \$1.5$$

- Relative Change (no units)

$$= \frac{(\text{new value}) - (\text{old value})}{(\text{old value})}$$

$$= \frac{1.5}{4} = \frac{3/2}{4} \cdot \frac{2}{2} = \frac{3}{8}$$

$$= 0.375 = 37.5\%$$

Ex: On July 7, (1999) the world record for fastest mile decreased by 0.56% to 223.13 seconds. What was the previous record?

Soln: Let x be the previous record time.

$$(\text{Relative Change}) = \frac{(\text{new value}) - (\text{old value})}{(\text{old value})}$$

$$-0.0056 = \frac{223.13 - x}{x}$$

$$-0.0056x = 223.13 - x$$

$$x - 0.0056x = 223.13$$

$$x(1 - 0.0056) = 223.13$$

$$x = \frac{223.13}{1 - 0.0056} = \boxed{224.39 \text{ seconds}}$$

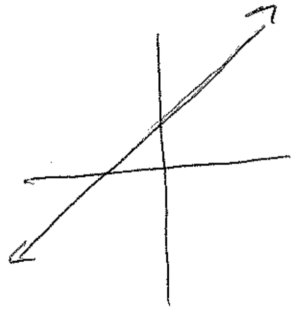
Concavity

- A function is concave up if the direction of the function bends upward as we move from left to right.
- A function is concave down if the direction of the function bends downward as we move from left to right.

Concave up

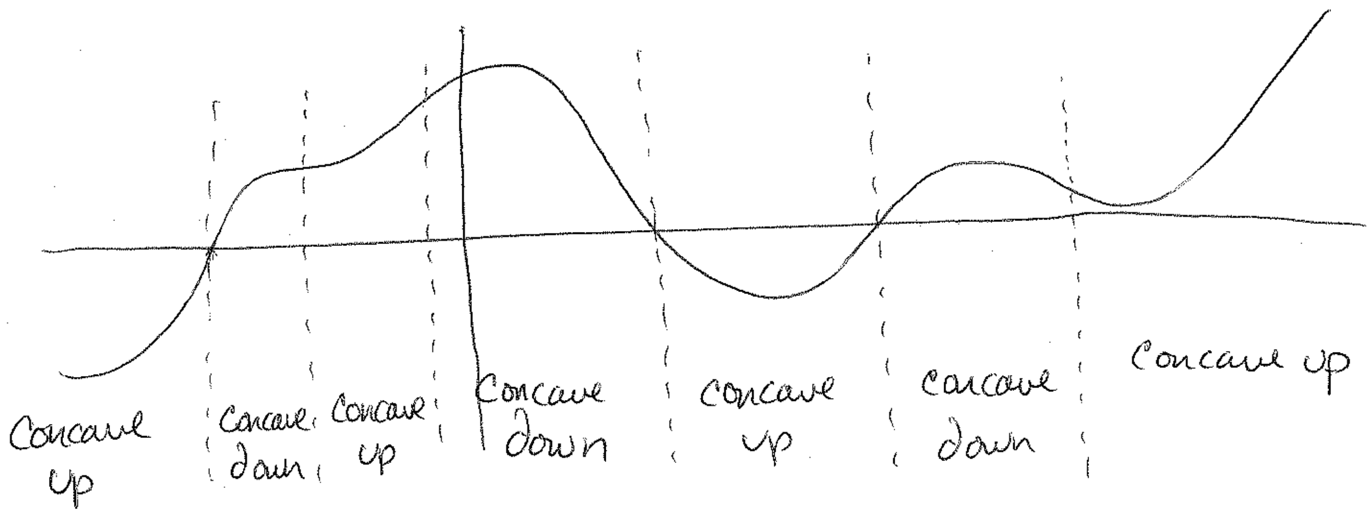
Concave down

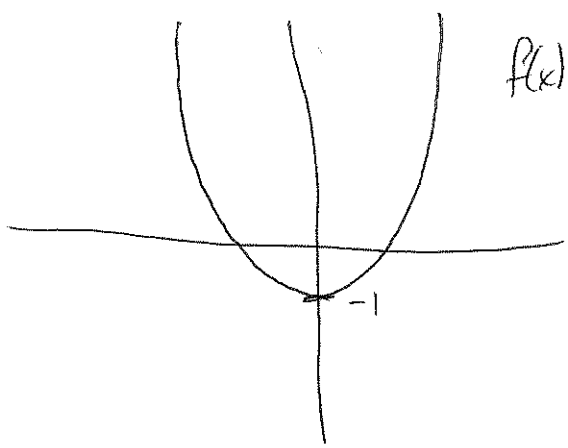
Note:



- A linear function is neither concave up nor concave down.

Ex





$f(x) = x^2 - 1$

Concave up everywhere

- A function is <u>concave up</u> where the average rate of change of a small interval increases as the interval moves from left to right.

- A function is <u>concave down</u> where the avg. rate of change of a small interval decreases as the interval moves from left to right.

Ex:

x	1	2	5	7	8
$f(x)$	34	58	130	178	202
	24	24	24	24	

x	1	2	3	5	6
$g(x)$	50	70	90	125	141
	20	20	17.5	16	

- Classify the concavity of these functions; if one of these functions is linear, give a formula.

Soln

- Avg rate of change of f from $x=1$ to $x=2$: $\dfrac{f(2)-f(1)}{2-1}$

$$= \frac{58-34}{1} \frac{\text{dollars}}{\text{meters}}$$

$$= \boxed{24 \text{ dollars per meter}}$$

- " " " from $x=2$ to $x=5$: $\dfrac{f(5)-f(2)}{5-2}$

$$= 24$$

Announcements

- Books available at Addams, SC Bookstore
- Quiz #1 – in class – Monday 1.1, 1.2, 1.3

1.2 #13, $\boxed{\#19}$, #23, #32, #33

1.3 #5

1.2 #13: Company 1: \$40/day + \$0.15 per mile

" 2: \$50/day + \$0.10 per mile

(a) Prize for company 1: $f(x)$

$$f(x) = 40 + 0.15x$$

Prize for Company 2: $g(x)$

$$g(x) = 50 + 0.10x$$

19.

q	3	4	5	6
p	15	12	9	6

(a) Find q as a linear function of p.

q: outputs $\approx y$

p: inputs $\approx x$

• $M = \frac{\Delta \text{output}}{\Delta \text{input}} = \frac{\Delta q}{\Delta p} = \frac{4-3}{12-15} = -\frac{1}{3}$

• Point slope form: $y - y_0 = m(x - x_0)$

$q - q_0 = m(p - p_0)$ where $(p_0, q_0) = (15, 3)$

$q - 3 = -\frac{1}{3}(p - 15)$

$q - 3 = -\frac{1}{3}p + 5$

$\boxed{q = -\frac{1}{3}p + 8}$

(b) Find p as a linear function of q

outputs $\approx y$

inputs $\approx x$

• $m = \frac{\Delta \text{output}}{\Delta \text{input}} = \frac{\Delta p}{\Delta q} = \frac{-3}{1} = -3$

⋮

$q = -\frac{1}{3}p + 8$

$q - 8 = -\frac{1}{3}p$

$\boxed{-3q + 24 = p}$

Sec 1.4:

- The <u>cost function</u> $C(q)$ is the cost of producing q units of some good.
- The <u>revenue function</u> $R(q)$ is the total income received when q units are sold.

$$R(q) = (\text{price}) \cdot q$$

- The <u>profit function</u> $P(q)$ is the net income earned when q units are sold.

$$P(q) = R(q) - C(q)$$

Marginal Costs, Revenue, and Profit

- Marginal cost of the q^{th} unit: cost of producing q^{th} unit after $q-1$ units have been produced.

$$= C(q) - C(q-1)$$

- Marginal Revenue: extra income received for selling the q^{th} unit, above income received for selling the first $q-1$ $= R(q) - R(q-1)$
- Marginal Profit: $P(q) - P(q-1)$

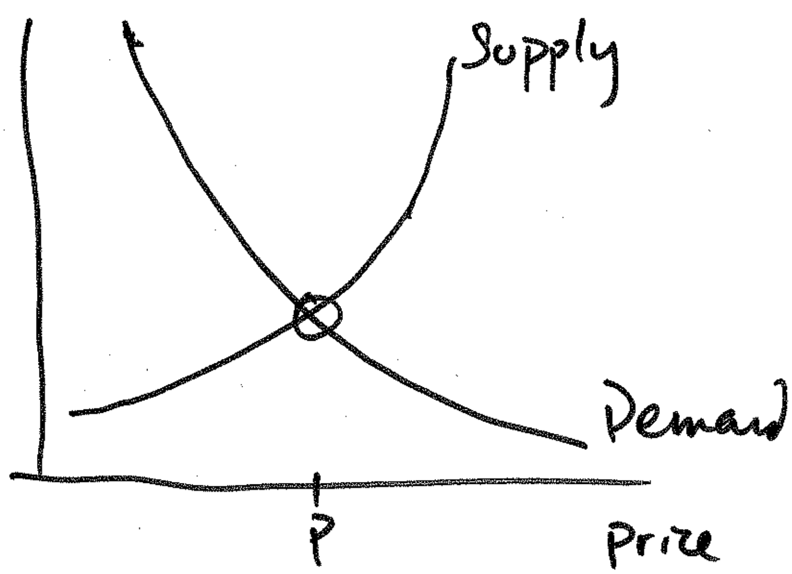
Supply and Demand curves

- Supply function: a function of price

  $S(p)$ = number of units that the market will produce at price p

- Demand function

  $D(p)$ = number of units the market demands when product has price p



Equilibrium point: $S(p) = D(p)$.

1.5 Exponential Functions

- If you increase the input by 1, the output gets multiplied by a constant.
- This constant is called the <u>base</u> (denoted a).

<u>Ex</u> $a = 2$

x	0	1	2	3	4
$g(x)$	1	2	4	8	16

- $g(x) = 2^x = \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{x \text{ copies}}$

when x is an integer.

Linear Functions

- If you increase the input by 1, the output will change (i.e. increase or decrease) by an additive constant.
- This constant is called the <u>slope</u>. (denoted m).

<u>Ex</u> $m = 2$

x	0	1	2	3	4
$f(x)$	1	3	5	7	9

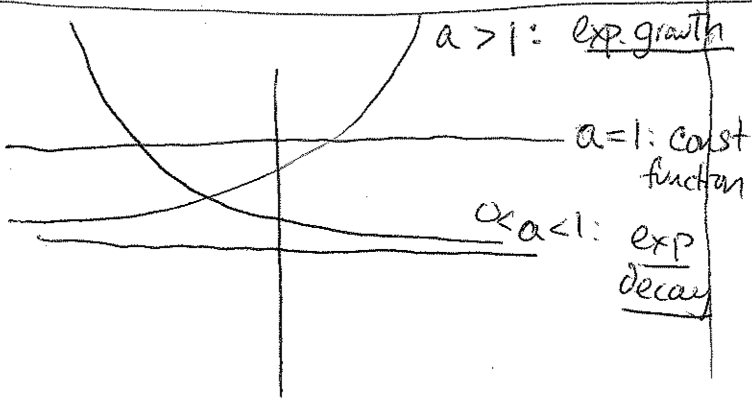
- $f(x) = 2x + 1$

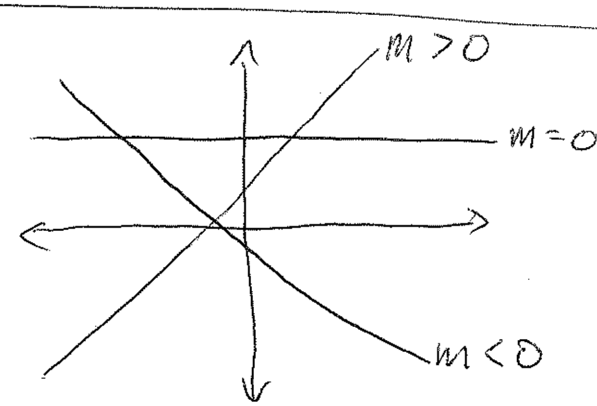
<u>Exp Functions</u>: constant ratios

$$\frac{g\cancel{f}(x)}{g\cancel{f}(x-1)} = \text{const}$$

<u>Linear functions</u>: constant difference

$$f(x) - f(x-1) = \text{const}$$





~~Ex~~ Supp

General form: $g(x) = k \cdot a^x$, where $a > 0$.

$f(x) = mx + b$

Ex Suppose a town starts with a population of 1000 and grows at a rate of 2% each year. Give a formula for the population $P(t)$ of the town after t years.

Soln: • Expect exp. growth; $a > 1$.

• Find base: what do we multiply by to get the population for next year?

Find with rel change formula

$$a = \frac{\text{Population after 1 year}}{\text{Initial population}} = \frac{1020}{1000}$$

$$= 1.02$$

$$= (1 + 0.02)$$

$$= \boxed{1 + r}$$

• $P(t) = k \cdot (1.02)^t$

$= 1000(1.02)^t$

Announcements

2

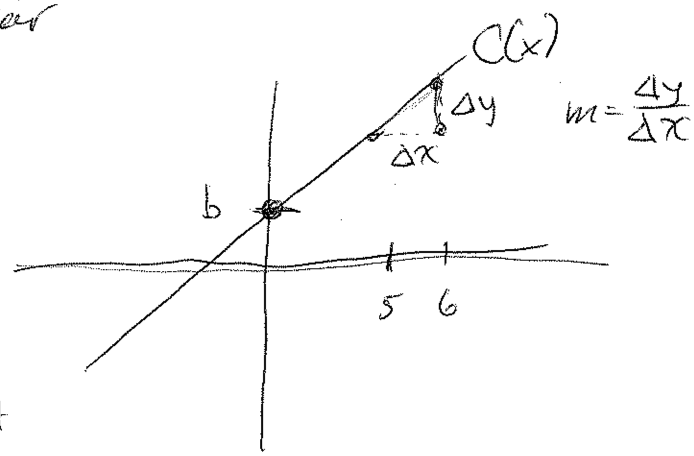
- Quiz #1 back Friday
- Hw #2 out now; 1.4–1.6; due Sept 7
- Quiz #2 at Friday; 1.4–1.6; due Sept 7
- Out of town Sept 5 – Sept 10; will still have email

1.4 $C(x)$ = cost function

= cost of producing x units of some good

→ If cost function is linear

$$C(x) = mx + b$$



- "variable cost" – How much does each unit cost
- "fixed cost" – How much does it cost to get started?

Exponential Functions

- If we're told that the relative change per unit time is r (in decimal form), then the function has the form

$$P(t) = P_0 \cdot (1+r)^t$$

Ex: An endangered species has 500 members at time $t=0$. Next year, there are only 450 members left. Suppose that the population continues to decrease exponentially. Give a formula for the size of population after t years.

Soln: • $P_0 = 500$

• $r = \frac{\text{new val} - (\text{old value})}{\text{old value}}$

$= \frac{450 - 500}{500} = \frac{-50}{500} = -\frac{1}{10} = -0.1$

• $P(t) = P_0(1+r)^t$

$= 500(1+(-0.1))^t$

$= \boxed{500(0.9)^t}$

• When does the population have size 1?

• Solve for t in: $1 = P(t)$

$1 = 500(0.9)^t$

$\frac{1}{500} = (0.9)^t$

$\ln\left(\frac{1}{500}\right) = \ln\left((0.9)^t\right)$

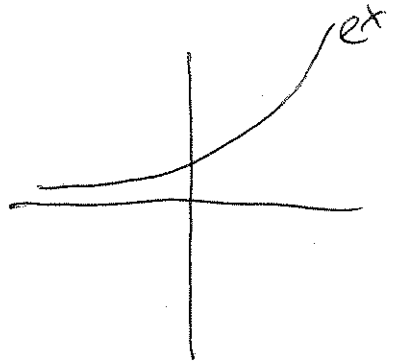
$\ln\left(\frac{1}{500}\right) = t\ln(0.9)$

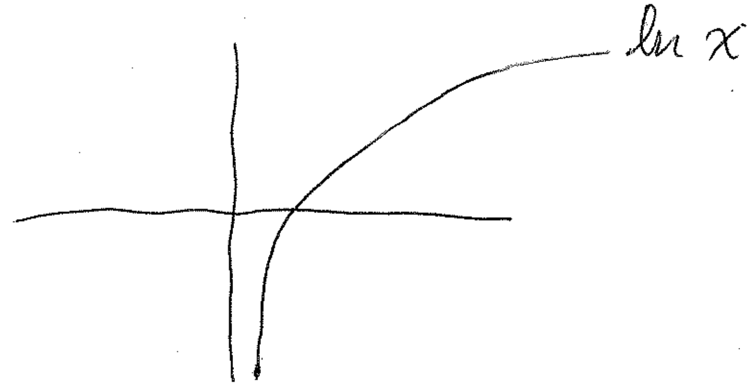
$\frac{\ln\left(\frac{1}{500}\right)}{\ln(0.9)} = t$

$\boxed{t = 58.98 \text{ years}}$

Properties of $\ln(x)$:

- Inverse of $e^x = (2.71828\ldots)^x$





- $e^{(\ln(x))} = x$
- $\ln(e^x) = x$
- $\ln(A \cdot B) = \ln(A) + \ln(B)$
- $\ln(A/B) = \ln(A) - \ln(B)$
- $\ln(A^B) = B \cdot \ln(A)$

Ex Solve for t: $12 = 3e^{2t} + 3$

$$9 = 3e^{2t}$$

$$3 = e^{2t}$$

$$\ln(3) = \ln(e^{2t})$$

$$\ln(3) = 2t$$

$$t = \frac{\ln(3)}{2} = 0.54\sim$$

Ex Town A has an initial population of 1,000 people and is growing at 4% each year.

Town B has init. pop. of 5 million and is declining at 2% each year.

When will the towns have the same population?

Soln

• $A(t) = A_0(1+r)^t$

$= 1000(1+0.04)^t = 1000(1.04)^t$

• $B(t) = B_0(1+r)^t$

$= 5{,}000{,}000(1-0.02)^t = 5{,}000{,}000(0.98)^t$

• Solve for t in $A(t) = B(t)$

$$\cancel{1000}(1.04)^t = 5000\cancel{000}(0.98)^t$$

$$\ln\left((1.04)^t\right) = \ln\left(5000 \cdot (0.98)^t\right)$$

Test

$$t\ln(1.04) = \ln(5000) + \ln\left((0.98)^t\right)$$

$$t\ln(1.04) = \ln(5000) + t\ln(0.98)$$

$$t\left(\ln(1.04) - \ln(0.98)\right) = \ln(5000)$$

$$t = \frac{\ln(5000)}{\ln(1.04) - \ln(0.98)} = \boxed{143.4 \text{ years}}$$

Two Forms of Exp Functions

$$P(t) = P_0 a^t \qquad \Big\| \qquad P(t) = P_0 e^{kt}$$

Ex:

$$P(t) = 1000 (1.4)^t$$

- $k = \ln(a)$
  $= \ln(1.4) \approx 0.336$

$$P(t) = 1000 \cdot e^{0.336 t}$$

- $a = e^k$

Notes: ~~a is~~ • k is called the continuous rate of change. ($k = 0.336$, or 33.6%)

- If $a = 1.4$, $r = 0.4$; rel rate of change 40%