1(a) Recall $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

$$\frac{d}{dx}\left(\sin^{-1}(xy)\right) = \frac{d}{dx}(x^2-3)$$

$$\frac{1}{\sqrt{1-(xy)^2}} \cdot \frac{d}{dx}(xy) = 2x$$

$$\frac{1}{\sqrt{1-(xy)^2}} \cdot \left(x\frac{dy}{dx} + 1\cdot y\right) = 2x$$

$$x\frac{dy}{dx} + y = 2x\sqrt{1-(xy)^2}$$

$$\boxed{\frac{dy}{dx} = \frac{2x\sqrt{1-(xy)^2} - y}{x}}$$

1(b). $$\frac{d}{dx}\left(y^x + 3\tan(y)\right) = \frac{d}{dx}(12)$$

$$\frac{d}{dx}(y^x) + \frac{d}{dx}(3\tan(y)) = 0.$$

$$\frac{d}{dx}(y^x) + 3\sec^2(y)\cdot\frac{dy}{dx} = 0.$$

We need to evaluate $\frac{d}{dx}(y^x)$. Here are two ways we can do this.

Method 1: Logarithmic Differentiation.

Let $f(x) = y^x$. Note $\left(\frac{d}{dx} f(x)\right) = f'(x)$.

$$\ln(f(x)) = x \ln(y).$$

$$\frac{d}{dx}\left(\ln(f(x))\right) = \frac{d}{dx}\left(x \ln(y)\right)$$

$$\frac{1}{f(x)} \cdot f'(x) = \ln(y) + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}$$

$$f'(x) = f(x)\left[\ln(y) + \frac{x}{y} \cdot \frac{dy}{dx}\right]$$

$$\left(\frac{d}{dx} f(x)\right) = y^x\left[\ln(y) + \frac{x}{y} \cdot \frac{dy}{dx}\right]$$

$$\left(\frac{d}{dx} y^x\right) = y^x\left[\ln(y) + \frac{x}{y} \cdot \frac{dy}{dx}\right]$$

Method 2: Rewrite $y^x = \left(e^{\ln(y)}\right)^x = e^{x\ln(y)}$.

$$\frac{d}{dx}(y^x) = \frac{d}{dx}\left(e^{x\ln(y)}\right)$$

$$= e^{x\ln(y)} \cdot \frac{d}{dx}\left(x \ln(y)\right)$$

$$= e^{x\ln(y)} \cdot \left(\ln(y) + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}\right)$$

$$= \left(e^{\ln(y)}\right)^x \cdot \left(\ln(y) + \frac{x}{y} \cdot \frac{dy}{dx}\right)$$

$$= y^x\left(\ln(y) + \frac{x}{y} \cdot \frac{dy}{dx}\right)$$

Continuing our computation,

$$\frac{d}{dx}(y^x) + 3\sec^2(y) \cdot \frac{dy}{dx} = 0$$

$$y^x\left(\ln(y) + \frac{x}{y}\cdot\frac{dy}{dx}\right) + 3\sec^2(y)\cdot\frac{dy}{dx} = 0$$

$$y^x\ln(y) + \frac{xy^x}{y}\frac{dy}{dx} + 3\sec^2(y)\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(\frac{xy^x}{y} + 3\sec^2(y)\right) = -y^x\ln(y)$$

$$\boxed{\frac{dy}{dx} = \frac{-y^x\ln(y)}{\frac{xy^x}{y} + 3\sec^2(y)}}.$$

2a. Both x^2 and $\sin(x)$ are continuous and differentiable everywhere. Therefore $x^2 + \sin(x)$ is continuous and differentiable everywhere. So, $\boxed{\text{MVT applies}}$. We conclude there exists c, $0 \le c \le \pi/2$ such that

$$f'(c) = \frac{f(\pi/2) - f(0)}{\pi/2 - 0} = \frac{(\pi/2)^2 + \sin(\pi/2) - (0^2 + \sin(0))}{\pi/2}$$

$$= \frac{(\pi/2)^2}{\pi/2} + \frac{1}{\pi/2}$$

$$= \pi/2 + \frac{2}{\pi}$$

Note: we are not asked to find c in this problem.

2b. $\boxed{\text{MVT does not apply}}$, because $\tan(\pi/2)$ is undefined, so $f(x)$ is undefined at $x = \pi/2$, and thus $f(x)$ is not continuous on $[0, \pi/2]$.

2c. $\boxed{\text{MVT does not apply}}$ because $f(x)$ is not defined at $x=-3$, so $f(x)$ is not continuous on $[-5,5]$.

2d. $\boxed{\text{MVT does not apply}}$, because $f(x)$ is not differentiable at $x=0$.

Note: $f(x)$ is continuous everywhere.

3. (a). $f'(x) = 5x^4 + 3x^2$. Because for all x and all even n, x^n is at least 0, for all x, $f'(x) \geq 0$.

$\boxed{\text{Therefore } f \text{ is increasing.}}$

(b) $f'(x) = 4x^3 + 4x = 4x(x^2+1)$.

Note that for all x, $x^2+1 \geq 1$. Therefore when $x > 0$,

$$f'(x) = \underbrace{4x}_{\text{positive}}\,\underbrace{(x^2+1)}_{\text{positive}}$$

$$> 0$$

so $f'(x) > 0$. When $x < 0$,

$$f'(x) = \underbrace{4x}_{\text{negative}}\,\underbrace{(x^2+1)}_{\text{positive}}$$

$$< 0$$

so $f'(x) < 0$.

Therefore $f(x)$ is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$, so $f(x)$ is $\boxed{\text{neither}}$ increasing nor decreasing everywhere.

4. Let $f(x) = x^4 + 6x^2 - 1$.

<u>$f(x)=0$ has at most 2 solutions:</u> If $f(x)=0$ had ≥ 3 solutions,

See Thm 9.3 p. 229

then because f is differentiable everywhere, $f'(x)=0$ would have to have at least 2 solutions. But $f'(x) = 4x^3 + 12x = 4x(x^2+12)$

and

no solution

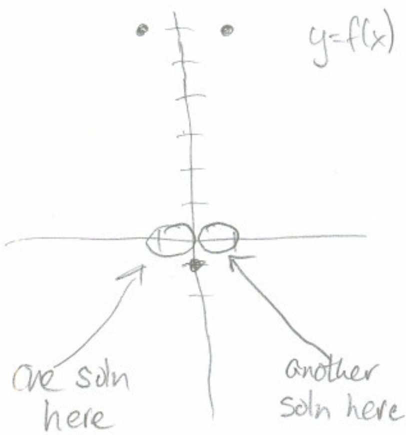
$$4x(x^2+12) = 0 \iff 4x = 0 \text{ or } x^2+12=0$$

$$\iff 4x = 0$$

$$\iff x = 0$$

so $f'(x)=0$ only has one solution.

<u>$f(x)=0$ has at least 2 solutions:</u> Note that $f(-1)=6$, $f(0)=-1$, $f(1)=6$.



Hence, by the Intermediate Value Theorem, there exists c_1, $-1 \leq c_1 \leq 0$ such that $f(c_1)=0$ and c_2, $0 \leq c_2 \leq 1$, such that $f(c_2)=0$. Therefore $f(x)=0$ has at least 2 solutions.

#5,6 are challenge problems; come talk to me if you are curious about them.