Math 221 DD4/DD5 Exam 3 Review Solns 30 Nov 2007 (1)

3.8

1.

t: time in min

r: radius in ft

A: area in ft^2.

- $\frac{dr}{dt} = 5$

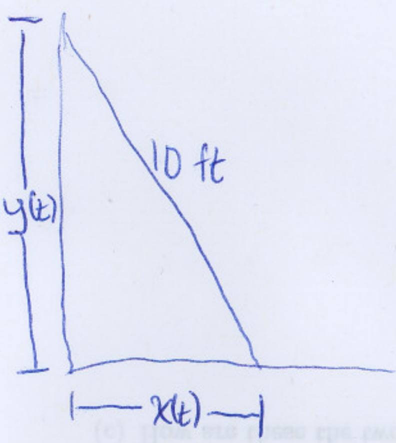
$$A(t) = \pi (r(t))^2$$

$$\frac{dA}{dt} = \pi(2r(t)) \cdot \frac{dr}{dt}$$

when $r(t) = 200$ ft, we have $\frac{dA}{dt} = \pi(2 \cdot 200) \cdot 5$

$$= \boxed{2000\pi \ ft^2/s}$$

2.



- t in seconds
- $x(t)$ in ft
- $y(t)$ in ft.
- $\frac{dx}{dt} = 3$

$$(x(t))^2 + (y(t))^2 = 10^2$$

$$\frac{d}{dt}\left((x(t))^2 + (y(t))^2\right) = \frac{d}{dt}(100)$$

$$2x(t) \cdot \frac{dx}{dt} + 2y(t)\frac{dy}{dt} = 0$$

When $x(t) = 6$, we have $(y(t))^2 = 100 - 6^2 = 64$, so $y(t) = 8$.

[illegible] (2)

Hence, when $x(t)=6$:

$$2\cdot 6\cdot 3 + 2\cdot 8\cdot \frac{dy}{dt} = 0$$

$$2\cdot 8\,\frac{dy}{dt} = -2\cdot 6\cdot 3$$

$$\frac{dy}{dt} = -\frac{6\cdot 3}{8} = -\frac{9}{4}$$

So the ladder drops at a rate of $\boxed{9/4 \text{ ft/s}}$.

<u>4.1</u> 1(a) $\displaystyle\int \frac{x + 2x^{3/4}}{x^{5/4}}\,dx = \int \frac{x}{x^{5/4}} + 2\frac{x^{3/4}}{x^{5/4}}\,dx$

$$= \int x^{-1/4} + 2x^{-1/2}\,dx$$

$$= \boxed{\frac{4}{3}x^{3/4} + 4x^{1/2} + C}$$

(b) $\displaystyle\int \frac{4}{\sqrt{1-x^2}}\,dx = \boxed{4\sin^{-1}(x) + C}$

(c) $\displaystyle\int \frac{e^x}{e^x+3}\,dx$ $\qquad u = e^x+3 \;\|\; du = e^x\,dx$

$$= \int \frac{1}{e^x+3}\cdot e^x\,dx$$

$$= \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|e^x+3| + C}$$

2. $f''(x) = 2x$ implies that $f'(x) = x^2 + a$, also where a is some const. Use $f'(0) = -3$ to solve for a:

$$f'(x) = x^2 + a$$

$$-3 = (0)^2 + a$$

$$a = -3.$$

Therefore $f'(x) = x^2 - 3$. Hence, $f(x) = \frac{1}{3}x^3 - 3x + b$, where b is a const. Use $f(0) = 2$ to solve for b:

$$f(x) = \frac{1}{3}x^3 - 3x + b$$

$$2 = \frac{1}{3}(0)^3 - 3 \cdot 0 + b$$

$$b = 2.$$

Hence, $\boxed{f(x) = \frac{1}{3}x^3 - 3x + 2}$.

3. $f'''(x) = \sin(x) - e^x$;

$$f''(x) = \int \sin(x) - e^x \, dx = -\cos(x) - e^x + a$$

$$f'(x) = \int -\cos(x) - e^x + a \, dx = -\sin(x) - e^x + ax + b$$

$$f(x) = \int -\sin(x) - e^x + ax + b \, dx = \cos(x) - e^x + \frac{a}{2}x^2 + bx + C$$

So $\boxed{f(x) = \cos(x) - e^x + \frac{a}{2}x^2 + bx + C}$ where a, b, c are arbitrary constants.

(Note: $f(x) = \cos(x) - e^x + ax^2 + bx + C$ is also a valid answer.)

4.2

1. $$\sum_{i=1}^{250} i^2 + 8 = \sum_{i=1}^{250} i^2 + \sum_{i=1}^{250} 8$$

$$= \frac{250 \cdot 251 \cdot (2 \cdot 250 + 1)}{6} + 8 \cdot 250$$

$$= \boxed{5{,}241{,}625}$$

2. $$\sum_{i=1}^{n} \frac{1}{n}\left[4\left(\frac{2i}{n}\right)^2 - \left(\frac{2i}{n}\right)\right] = \frac{1}{n}\sum_{i=1}^{n}\left(4\frac{4i^2}{n^2} - \frac{2i}{n}\right)$$

$$= \frac{1}{n}\left(\frac{16}{n^2}\sum_{i=1}^{n} i^2 - \frac{2}{n}\sum_{i=1}^{n} i\right)$$

$$= \frac{1}{n}\left(\frac{16}{n^2}\,\frac{n(n+1)(2n+1)}{6} - \frac{2}{n}\cdot\frac{n(n+1)}{2}\right)$$

$$= \frac{1}{n}\left(\frac{8(n+1)(2n+1)}{3n} - (n+1)\right)$$

$$= \frac{8}{3}\cdot\frac{2n^2+3n+1}{n^2} - \frac{n+1}{n}$$

$$\lim_{n\to\infty} \frac{8}{3}\cdot\frac{2n^2+3n+1}{n^2} - \frac{n+1}{n} = \lim_{n\to\infty} \frac{8}{3}\left(2 + \frac{3}{n^2} + \frac{1}{n^2}\right) - \left(1 + \frac{1}{n}\right)$$

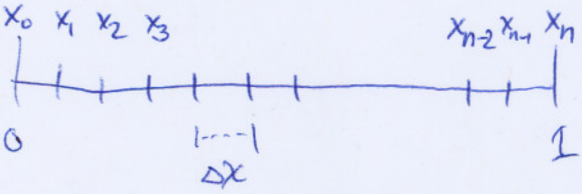
$$= \frac{8}{3}(2+0+0) - (1+0)$$

$$= \frac{16}{3} - 1$$

$$= \boxed{\frac{13}{3}}$$

4.3

1. $y = x^2 + 3x$ on $[0,1]$.



- $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$

- $x_i = x_0 + i \cdot \Delta x$
  $= 0 + i \cdot \frac{1}{n}$

Use right endpt evaluation
- $c_i = x_i = \dfrac{i}{n}$

Riemann Sum: $\displaystyle \sum_{i=1}^{n} f(c_i)\Delta x$

$$= \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n}$$

$$= \sum_{i=1}^{n} \left[\left(\frac{i}{n}\right)^2 + 3\frac{i}{n}\right] \cdot \frac{1}{n}$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} \frac{i^2}{n^2} + \sum_{i=1}^{n} \frac{3}{n} \cdot i \right)$$

$$= \frac{1}{n}\left(\frac{1}{n^2} \sum_{i=1}^{n} i^2 + \frac{3}{n} \sum_{i=1}^{n} i \right)$$

$$= \frac{1}{n}\left(\frac{1}{n^2} \frac{n(n+1)(2n+1)}{6} + \frac{3}{n} \frac{n(n+1)}{2} \right)$$

$$= \frac{\cancel{n}(n+1)(2n+1)}{6n^2} + \frac{3}{2}\frac{n+1}{n}$$

$$= \frac{2n^2+3n+1}{6n^2} + \frac{3}{2}\frac{n+1}{n}$$

$$\lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} + \frac{3}{2}\frac{n+1}{n} = \lim_{n\to\infty} \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} + \frac{3}{2}\left(1+\frac{1}{n}\right)$$

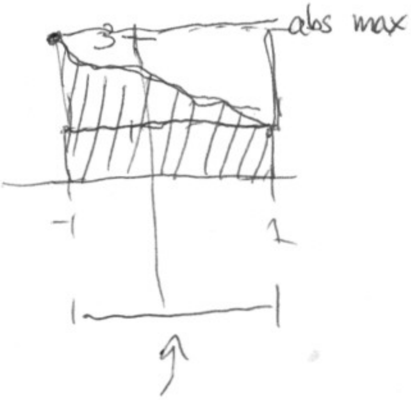
$$= \frac{1}{3} + 0 + 0 + \frac{3}{2}(1+0) = \frac{1}{3} + \frac{3}{2} =$$

2. $\boxed{\text{True}}$

4.4 1. IMVT: If f is continuous on $[a,b]$, then there is c, $a<c<b$, such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

2.



width $b-a=2$

Find abs min, abs max of $f(x) = \frac{3}{x^3+2}$ on $[-1, 1]$:

- Critical pts: $f(x) = 3(x^3+2)^{-1}$

$$f'(x) = -3(x^3+2)^{-2}\cdot 3x^2$$

$$= -9\cdot\frac{x^2}{(x^3+2)^2}$$

So $f'(x)=0$ implies

$$-9\cdot\frac{x^2}{(x^3+2)^2} = 0$$

$$x^2 = 0$$

$$x = 0.$$

- Candidates: $x=-1$, $x=0$, $x=1$

abs max $\Longrightarrow$ · $f(-1) = \frac{3}{(-1)+2} = 3 = M$

· $f(0) = \frac{3}{0+2} = \frac{3}{2}$

abs min. $\Longrightarrow$ · $f(1) = \frac{3}{1+2} = 1 = m.$

So the area of shaded region is between $m\cdot 2$ and $M\cdot 2$.

Therefore $\boxed{2 = 1\cdot 2 \le \int_{-1}^{1}\frac{3}{x^3+2}\,dx \le 3\cdot 2 = 6}$

1(a). FTC(I):

If f is continuous on $[a,b]$ and $F(x)$ is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

(b) FTC(II):

If f is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, then

$$F'(x) = f(x)$$

on $[a,b]$.

2(a). $$\int_0^2 (\sqrt{x}+1)^2\,dx = \int_0^2 x + \overset{2x^{\frac{1}{2}}}{2\sqrt{x}} + 1\,dx$$

$$= \left(\frac{x^2}{2} + \frac{4}{3}x^{3/2} + x\right)\Big|_0^2$$

$$= \frac{4}{2} + \frac{4}{3}(2)^{3/2} + 2 - 0$$

$$= 4 + \frac{4}{3}\sqrt{8} = \boxed{4 + \frac{8}{3}\sqrt{2}}$$

(b) $$\int_0^{\pi/3} \frac{3}{\cos^2(x)}\,dx = 3\int_0^{\pi/3} \sec^2(x)\,dx$$

$$= 3\left(\tan(x)\right)\Big|_0^{\pi/3}$$

$$= 3\left(\tan(\pi/3) - \tan(0)\right) = 3\left(\sqrt{3} - 0\right) = \boxed{3\sqrt{3}}$$

(c) $$\int_1^2 \frac{x^2 - 3x + 4}{x^2}\,dx = \int_1^2 1 - 3\cdot\frac{1}{x} + 4x^{-2}\,dx$$

$$= \left(x - 3\ln|x| - 4x^{-1}\right)\Big|_1^2$$

$$= \left(2 - 3\ln(2) - \frac{4}{2}\right) - \left(1 - 3\ln(1) - \frac{4}{1}\right)$$

$$= -\ln(8) \qquad - (-3 - 3\cdot 0)$$

$$= \boxed{3 - \ln(8)}$$

3. $f(x) = \int_{x^3}^{x^2} \sin(\sqrt{t})\,dt$

$$= \int_0^{x^2} \sin(\sqrt{t})\,dt - \int_0^{x^3} \sin(\sqrt{t})\,dt$$

By FTC(II):

Let $u(x) = x^2$, $\quad g(x) = \int_0^x \sin(\sqrt{t})\,dt$ $\quad\Big\|\quad u'(x) = 2x$ $\quad g'(x) = \sin(\sqrt{x})$

$v(x) = x^3$ $\quad\quad v'(x) = 3x^2$

Note: $f(x) = g(u(x)) - g(v(x))$

$$\cancel{f'(x) = g'(u(x))}$$

$$f'(x) = g'(u(x))\cdot u'(x) - g'(v(x))\cdot v'(x)$$

$$= \sin\left(\sqrt{u(x)}\right)\cdot 2x - \sin\left(\sqrt{v(x)}\right)\cdot v'(x)$$

$$= \sin\left(\sqrt{x^2}\right)\cdot 2x - \sin\left(\sqrt{x^3}\right)\cdot 3x^2$$

$$= \boxed{2x\sin(x) - 3x^2\sin\left(x^{3/2}\right)}$$

4.6 (a) $\int \sin^3(x)\cos(x)\,dx$

$\cdot\, u(x) = \sin(x)$
$\cdot\, du = \cos(x)\,dx$

$$= \int u^3 \quad du$$

$$= \frac{u^4}{4} + C$$

$$= \boxed{\frac{\sin^4(x)}{4} + C}$$

(b) $\int \frac{x}{x^2+4}\,dx$

$\cdot\, u(x) = x^2+4$
$\cdot\, du = 2x\,dx$

$$= \frac{1}{2}\int \frac{1}{x^2+4} \cdot 2x\,dx$$

$$= \frac{1}{2}\int \frac{1}{u} \quad du$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \boxed{\frac{1}{2}\ln|x^2+4| + C}$$

(c) $\int \frac{2x+3}{x+7}\,dx$

$u = x+7, \quad x = u-7$
$du = dx$

$$= \int \frac{2(u-7)+3}{u}\,du$$

$$= \int \frac{2u - 11}{u}\,du$$

$$= \int 2 - 11/u \, du$$

$$= 2u - 11\ln|u| + C = \boxed{2(x+7) - 11\ln|x+7| + C}$$

Note: $\boxed{2x - 11\ln|x+7| + C}$ is also a valid answer.

(d) $\int \frac{4}{x(\ln(x)+1)^2} dx$

$u(x) = \ln(x) + 1$
$du = \frac{1}{x} dx$

10

$= \int \frac{4}{(\ln(x)+1)^2} \cdot \frac{1}{x} dx$

$= \int \frac{4}{u^2} \, du$

$= \int 4u^{-2} \, du = -4u^{-1} + C$

$= -\frac{4}{u} + C = \boxed{-\frac{4}{\ln(x)+1} + C}$

2(a) $\int_1^e \frac{\ln(x)}{x} dx$

$u(x) = \ln(x)$
$du = \frac{1}{x} dx$

$= \int_1^e \ln(x) \cdot \frac{1}{x} dx$

$= \int_{u(1)}^{u(e)} u \, du$

$= \int_0^1 u \, du$

$\begin{cases} u(e) = \ln(e) = 1 \\ u(1) = \ln(1) = 0 \end{cases}$

$= \left. \frac{u^2}{2} \right|_0^1$

$= \boxed{\frac{1}{2}}$

(b) $\int_0^2 x\sqrt{x^2+1} \, dx$

$u(x) = x^2 + 1$
$du = 2x \, dx$

$= \frac{1}{2} \int_0^2 \sqrt{x^2+1} \cdot 2x \, dx$

$= \frac{1}{2} \int_{u(0)}^{u(2)} \sqrt{u} \, du$

$$= \frac{1}{2}\int_1^5 u^{1/2}\, du$$

$$= \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^5$$

$$= \frac{1}{2}\left(\frac{2}{3}\cdot 5^{3/2} - \frac{2}{3}\cdot 1^{3/2}\right)$$

$$= \boxed{\frac{1}{3}5^{3/2} - \frac{1}{3}}$$

(c) $\int_0^2 x\sqrt{x+1}\, dx$ $\quad$ • $u(x) = x+1$ $\;||\;$ $x = u-1$

• $du = dx$

$$= \int_{u(0)}^{u(2)} (u-1)\sqrt{u}\, du$$

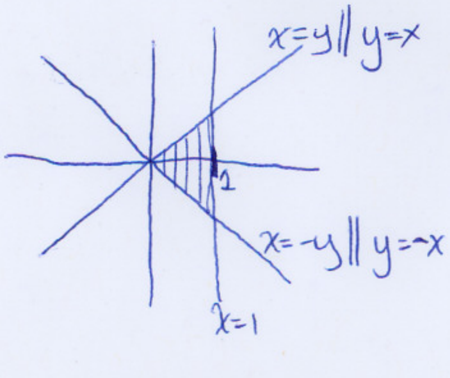
$$= \int_1^3 u\sqrt{u} - \sqrt{u}\, du$$

$$= \int_1^3 u^{3/2} - u^{1/2}\, du = \left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Big|_1^3$$

$$= \left(\frac{2}{5}\cdot 3^{5/2} - \frac{2}{3}3^{3/2}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)$$

$$= \boxed{\frac{2}{5}3^{5/2} - \frac{2}{3}3^{3/2} + \frac{4}{15}}$$

5.1 $\quad$ 1.



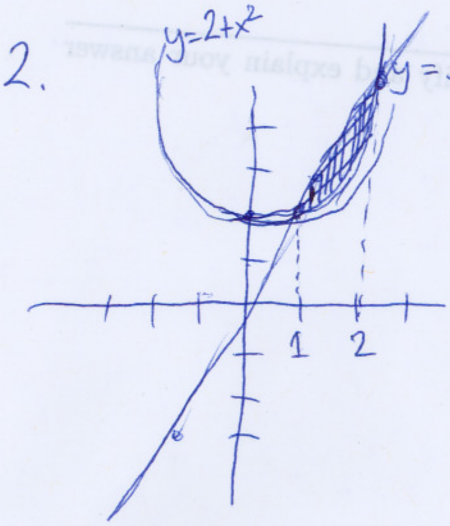
$$A = \int_0^1 x - (-x)\, dx.$$

$$= \int_0^1 2x\, dx$$

$$= x^2\Big|_0^1$$

$$= \boxed{1}$$

2.



Points of intersection:

$$2+x^2=3x$$
$$x^2-3x+2=0$$
$$(x-2)(x-1)=0$$
$$x=1,2$$

$$A=\int_1^2 (3x)-(2+x^2)\,dx$$

$$=\int_1^2 -x^2+3x-2\,dx$$

$$=\left(-\tfrac{1}{3}x^3+\tfrac{3}{2}x^2-2x\right)\Big|_1^2$$

$$=\left(-\tfrac{1}{3}\cdot 8+\tfrac{3}{2}\cdot 4-2\cdot 2\right)-\left(-\tfrac{1}{3}+\tfrac{3}{2}-2\right)$$

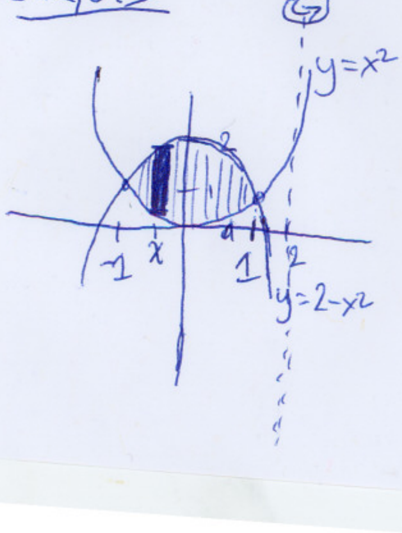
$$=\left(-\tfrac{8}{3}+6-4\right)-\left(-\tfrac{2}{6}+\tfrac{9}{6}-\tfrac{12}{6}\right)$$

$$=\left(2-\tfrac{8}{3}\right)-\left(-\tfrac{5}{6}\right)$$

$$=\frac{12}{6}-\frac{16}{6}+\frac{5}{6}$$

$$=\boxed{\frac{1}{6}}$$

5.2.5.3



- Integrate along x.
- Intersection pts: $x^2=2-x^2$

$$2x^2=2$$
$$x^2=1$$
$$x=\pm 1$$

- Use cylindrical shells:

$$Vol=\int_{-1}^{1} 2\pi\,(\text{radius})(\text{height})\,dx$$

$$= \int_{-1}^{1} 2\pi\,(2-x)(2-x^2-x^2)\,dx$$

$$= \int_{-1}^{1} 2\pi\,(2-x)(2-2x^2)\,dx$$

$$= \int_{-1}^{1} 4\pi\,(2-x)(1-x^2)\,dx$$

$$= 4\pi\int_{-1}^{1} 2-2x^2-x+x^3\,dx$$

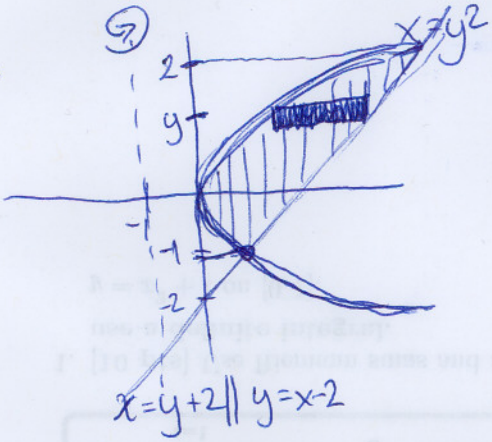
$$= 4\pi\left(2x-\tfrac{2}{3}x^3-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4\right)\Big|_{-1}^{1}$$

$$= 4\pi\left[\left(2-\tfrac{2}{3}-\tfrac{1}{2}+\tfrac{1}{4}\right)-\left(-2+\tfrac{2}{3}-\tfrac{1}{2}+\tfrac{1}{4}\right)\right]$$

$$= 4\pi\left[4-\tfrac{4}{3}\right]$$

$$= 4\pi\cdot\tfrac{8}{3} = \boxed{\tfrac{32}{3}\pi}$$

2.



- Integrate along y.
- Intersection pts:

$$y+2=y^2$$
$$y^2-y-2=0$$
$$(y-2)(y+1)=0$$
$$y=-1,2.$$

- Use washers.

$$\mathrm{Vol} = \int_{-1}^{2} \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2\,dy$$

$$= \int_{-1}^{2} \pi\left((y+2)-(-1)\right)^2 - \pi\left(y^2-(-1)\right)^2\,dy$$

$$= \int_{-1}^{2} \pi(y+3)^2 - \pi(y^2+1)^2\,dy$$

$$= \pi \int_{-1}^{2} (y+3)^2\, dy \quad - \pi \int_{-1}^{2} (y^2+1)^2\, dy$$

• Let $u(y) = y+3$

$du = dy$

$$= \pi \int_{u(-1)}^{u(2)} u^2\, du \quad - \pi \int_{-1}^{2} y^4 + 2y^2 + 1\, dy$$

$$= \pi \int_{2}^{5} u^2\, du \quad - \pi \left(\frac{y^5}{5} + \frac{2}{3} y^3 + y \right) \Big|_{-1}^{2}$$

$$= \pi \left(\frac{u^3}{3} \right) \Big|_{2}^{5} \quad - \pi \left(\left(\frac{32}{5} + \frac{2}{3} 8 + 2 \right) - \left(-\frac{1}{5} - \frac{2}{3} - 1 \right) \right)$$

$$= \pi \left(\frac{125}{3} - \frac{8}{3} \right) \quad - \pi \left(\frac{78}{5} \right)$$

$$= \boxed{\frac{117}{5} \pi}$$

General Review

(a) def The indefinite integral of $f(x)$ is defined to be

$$\int f(x)\, dx = F(x) + C$$

where $F(x)$ is an antiderivative of f and c is a constant.

(b) def The definite integral of $f(x)$ from a to b is the limit of Riemann sum

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x$$

whenever the limit exists and is the same for all choices of the evaluation points $c_1, c_2, \ldots, c_n$ in the intervals $[x_0, x_1], [x_1, x_2], \ldots [x_{n-1}, x_n]$

(c) The Fundamental Thm of Calculus provides the connection.

$\Longrightarrow$ Good Luck on Monday! $\Longleftarrow$