Math 221 DD4/DD5 Exam 2 Review Solns

1. $$f'(x) = \frac{d}{dx}\left(\left(2+\tan^{-1}(x)\right)^{1/2}\right)$$

$$= \frac{1}{2}\left(2+\tan^{-1}(x)\right)^{-1/2} \cdot \frac{d}{dx}\left(2+\tan^{-1}(x)\right)$$

$$= \boxed{\frac{1}{2}\left(2+\tan^{-1}(x)\right)^{-1/2} \cdot \frac{1}{1+x^2}}$$

2. Mean Value Theorem: If f is continuous on $[a,b]$ and differentiable on (a,b) then there is some point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

3. $\boxed{\text{False.}}$ ~~Just because~~ Even though f is continuous at $x=a$, f may not be differentiable at $x=a$.

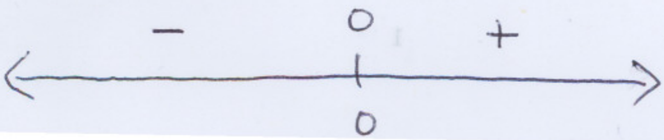
4. $f'(x) = 4x^3 + 4x$

$= 4x(x^2+1)$

Critical pts: $4x=0$ or $x^2=-1$

$x=0$ no soln

Sign Chart for f':



Therefore, $f'(x) < 0$ on $(-\infty, 0)$ and

$f'(x) > 0$ on $(0, \infty)$.

Hence f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

(Note that f has a local minimum at $x=0$.)

5. The linear approximation is given by the formula

$$L(x) = f'(x_0)(x - x_0) + f(x_0).$$

Compute: $f'(x) = \frac{1}{2}(2x+9)^{-\frac{1}{2}} \cdot 2 = \frac{1}{\sqrt{2x+9}}$

$$f'(x_0) = f'(0) = \frac{1}{\sqrt{2 \cdot 0 + 9}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

Compute: $f(x_0) = f(0) = \sqrt{2 \cdot 0 + 9} = \sqrt{9} = 3$

Therefore

$$L(x) = \frac{1}{3}(x - 0) + 3$$

$$= \boxed{\frac{x}{3} + 3}.$$

6. Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

~~In this~~ Newton's Method finds solutions to $f(x)=0$. We want solutions to $x^4-3x^2+1=0$, so set

$$f(x) = x^4 - 3x^2 + 1.$$

Next, we need $f'(x) = 4x^3 - 6x$.

Next, solve for x_1, given our first guess $x_0 = 1$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{f(1)}{f'(1)}$$

$$= 1 - \frac{1-3+1}{4-6} = 1 - \frac{-1}{-2} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Next, solve for x_2, given that $x_1 = \frac{1}{2}$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= \frac{1}{2} - \frac{f(\frac{1}{2})}{f'(\frac{1}{2})}$$

$$= \frac{1}{2} - \frac{(\frac{1}{2})^4 - 3(\frac{1}{2})^2 + 1}{4\cdot(\frac{1}{2})^3 - 6\cdot\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{(\frac{1}{2})^4 - 3(\frac{1}{2})^2 + 1}{4(\frac{1}{2})^3 - 6\frac{1}{2}} \cdot \frac{2^4}{2^4}$$

$$= \frac{1}{2} - \frac{1 - 3\cdot 2^2 + 2^4}{4\cdot 2 - 6\cdot 2^3}$$

$$= \frac{1}{2} - \frac{1 - 3\cdot 4 + 16}{8 - 6\cdot 8}$$

$$= \frac{1}{2} - \frac{5}{-40}$$

$$= \frac{1}{2} + \frac{1}{8} = \frac{4}{8} + \frac{1}{8} = \frac{5}{8}.$$

Therefore $\boxed{x_1 = \frac{1}{2} \text{ and } x_2 = \frac{5}{8}}$.

7. $$\lim_{x\to 0^+} \frac{\sin(x)}{\sqrt{x}}$$

(Note the form of the limit: $\sin(x) \to 0$

$\sqrt{x} \to 0$, so this limit has form $\frac{0}{0}$

and L'Hopital's rule applies.)

$$= \lim_{x\to 0^+} \frac{\frac{d}{dx}(\sin(x))}{\frac{d}{dx}(x^{\frac{1}{2}})}$$

$$= \lim_{x\to 0^+} \frac{\cos(x)}{\frac{1}{2}x^{-\frac{1}{2}}}$$

$$= \lim_{x\to 0^+} \frac{\cos(x)}{\frac{1}{2\,x^{\frac{1}{2}}}}$$

$$= \lim_{x\to 0^+} 2\cdot x^{\frac{1}{2}}\cdot \cos(x)$$

(Now Limit has form $0\cdot 1$

which means the limit is 0.)

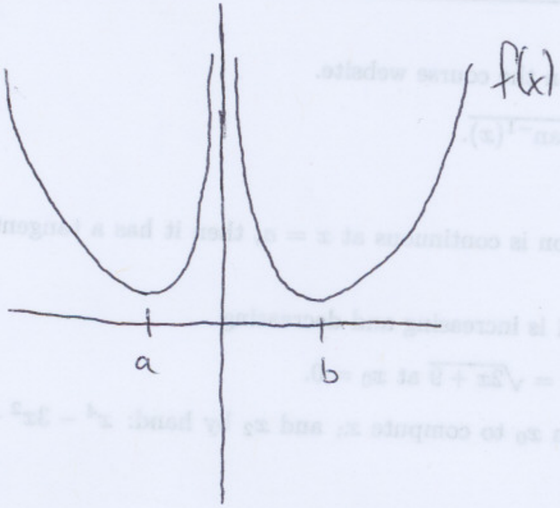
$$= \left(\lim_{x\to 0^+} 2\sqrt{x}\right)\cdot\left(\lim_{x\to 0^+} \cos(x)\right)$$

$$= 2\sqrt{0} \cdot \cos(0)$$

$$= 0 \cdot 1$$

$$= \boxed{0}$$

8.



f(x) has local minimums at $x=a$ and $x=b$, but no local maximums between $x=a$ and $x=b$. Note that f is not continuous on $[a,b]$ ($f(x)$ is discontinuous at $x=0$), and this must be the case, or else the Extreme Value Theorem tells us f will have an absolute maximum on $[a,b]$ (and therefore a local maximum also).

For a specific example, try $f(x) = \frac{e^{(x^2)}}{x^2}$.

9. $f'(x) = 2x e^{-4x} + x^2 \cdot e^{-4x} \cdot (-4)$

$= 2x e^{-4x} (1 - 2x)$

Note: f is cont. and diff. on $\mathbb{R}$.

Critical pts: $f'(x) = 0$

$2x = 0$ or $e^{-4x} = 0$ or $1 - 2x = 0$

$x = 0$ no soln $x = \frac{1}{2}$

So critical points are $\{0, \frac{1}{2}\}$.

(a) $[-2, 0]$

$f(-2) = (-2)^2 e^{-4 \cdot (-2)} = 4e^8 \leftarrow$ abs max

$f(0) = 0^2 \cdot e^{-4 \cdot 0} = 0 \leftarrow$ abs min

So ~~there~~ f(x) has an abs. max on $[-2, 0]$ at $(-2, 4e^8)$ and an abs. min on $[-2, 0]$ at $(0, 0)$.

(b) $[0, 4]$

$f(0) = 0 \leftarrow$ abs min

$f(\frac{1}{2}) = (\frac{1}{2})^2 e^{-4 \cdot \frac{1}{2}} = \frac{1}{4} e^{-2} = \frac{1}{4e^2} \approx 0.034 \leftarrow$ abs max

$f(4) = 4^2 \cdot e^{-4 \cdot 4} = 16 e^{-16} = \frac{16}{e^{16}} \approx 0.0000018$

So, on $[0,4]$, $f(x)$ has an absolute maximum at $\left(\frac{1}{2}, \frac{1}{4e^2}\right)$ and an absolute minimum at $(0,0)$.

10. <u>Critical points:</u> places where $f(x)$ is defined and $f'(x)=0$ or $f'(x)$ DNE.

$$f(x) = \frac{\sqrt{x}}{1+\sqrt{x}} = \frac{x^{\frac{1}{2}}}{1+x^{\frac{1}{2}}} \qquad \text{domain}(f) = [0, \infty)$$

$$f'(x) = \frac{(1+x^{\frac{1}{2}})\frac{1}{2}x^{-\frac{1}{2}} - x^{\frac{1}{2}}\cdot\frac{1}{2}x^{-\frac{1}{2}}}{(1+x^{\frac{1}{2}})^2}$$

$$= \frac{\frac{1+x^{\frac{1}{2}}}{2\sqrt{x}} - \frac{1}{2}}{(1+\sqrt{x})^2} \cdot \frac{2\sqrt{x}}{2\sqrt{x}}$$

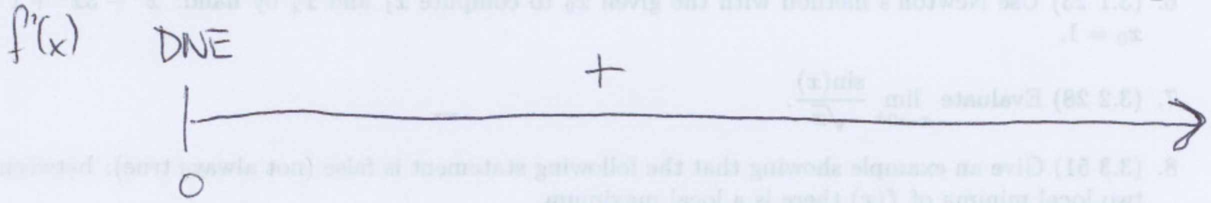
$$= \frac{1+\sqrt{x} - \sqrt{x}}{(1+\sqrt{x})^2 \cdot 2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}(1+\sqrt{x})^2} \qquad \text{domain}(f') = (0, \infty).$$

Note that $f'(x)=0$ has no soln, but $f'(0)$ DNE,

so $\boxed{f(x) \text{ has one critical point at } x=0.}$

Local Extrema: Use first derivative test. Here is a sign chart for $f'(x)$:



Note $f'(x) > 0$ for all $x > 0$, so f is increasing on $(0, \infty)$. Because f is continuous on $[0, \infty)$,

$$\boxed{f \text{ has a local minimum at } x=0;\ \text{i.e. } (0,0).}$$

Inflection points: First find points where $f''(x) = 0$ or $f''(x)$ DNE.

$$f'(x) = \frac{1}{2\sqrt{x}(1+\sqrt{x})^2} = \left(2\sqrt{x}(1+\sqrt{x})^2\right)^{-1}$$

$$f''(x) = -\left(2\sqrt{x}(1+\sqrt{x})^2\right)^{-2} \cdot \left(2 \cdot \tfrac{1}{2} \cdot x^{-\frac{1}{2}} \cdot (1+\sqrt{x})^2 + 2\sqrt{x} \cdot 2(1+\sqrt{x}) \cdot \tfrac{1}{2}\right.$$

$$= -\frac{x^{-\frac{1}{2}}(1+\sqrt{x})^2 + 2 \cdot \sqrt{x} \cdot x^{-\frac{1}{2}}(1+\sqrt{x})}{\left(2\sqrt{x}(1+\sqrt{x})^2\right)^2}$$

⑩

$$= -\frac{x^{-\frac{1}{2}}(1+\sqrt{x})^2+2(1+\sqrt{x})}{4x(1+\sqrt{x})^4}\cdot\frac{\frac{1}{1+\sqrt{x}}}{\frac{1}{1+\sqrt{x}}}$$

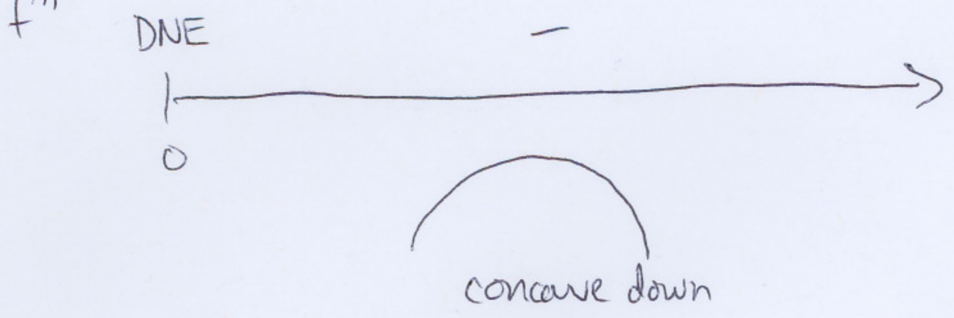
$$= -\frac{x^{-\frac{1}{2}}(1+\sqrt{x})+2}{4x(1+\sqrt{x})^3}\cdot\frac{\sqrt{x}}{\sqrt{x}}$$

$$= -\frac{(1+\sqrt{x})+2\sqrt{x}}{4x^{3/2}(1+\sqrt{x})^3}$$

$$= -\frac{3\sqrt{x}+1}{4x^{3/2}(1+\sqrt{x})^3}$$

domain $(f'') = (0, \infty)$.

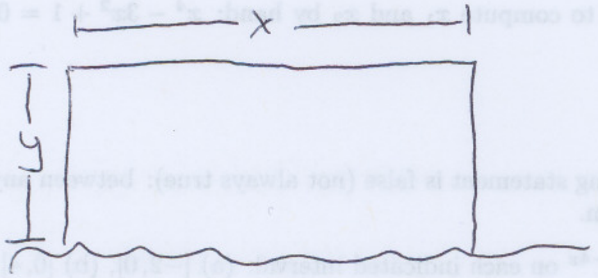
Now we make a sign chart for f'':



Note $f''(x) < 0$ for all x in $(0, \infty)$, so $f(x)$ is concave down on $(0, \infty)$. Because the concavity of f never changes, $\boxed{\text{f has no inflection points}}$

Note: ~~That This problem~~ Problem 10 is too long and difficult to appear on the Exam.

11.



- 96 ft of fencing $\Rightarrow x + 2y = 96$

$2y = 96 - x$

$y = 48 - \frac{1}{2}x$

- Area $= x \cdot y$

$= x \cdot \left(48 - \frac{1}{2}x\right)$

$= 48x - \frac{1}{2}x^2$.

Let $f(x) = -\frac{1}{2}x^2 + 48x$. We must find the absolute maximum of $f(x)$ for x in the range $[0, 96]$.

$f'(x) = -x + 48$

$f'(x) = 0 \Rightarrow -x + 48 = 0 \Rightarrow x = 48$.

So just one critical pt: $\{48\}$.

Compute $f(0) = 0$

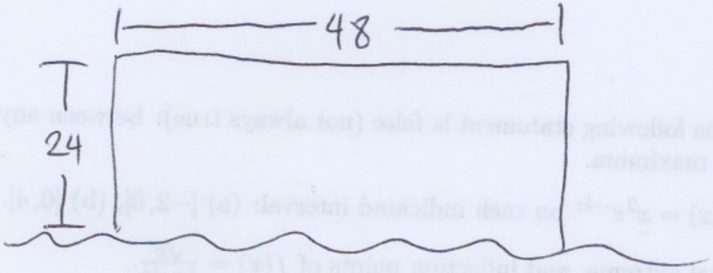
$f(48) = -\frac{1}{2}(48)^2 + 48 \cdot 48 = (48)^2\left(1 - \frac{1}{2}\right) = \frac{(48)^2}{2}$

$= 1152$

$f(96) = 96 \cdot y = 96 \cdot 0 = 0$.

Therefore the abs max of f on $[0, 96]$ occurs at $x = 48$. Hence, we should build our fence with

$$\boxed{x = 48 \quad \text{and} \quad y = 48 - \tfrac{1}{2}\cdot 48 = 48/2 = 24.}$$



Area: $1152\ ft^2$,

Good Luck on the Exam!