Math 221 DD4/DD5 Exam 1 Review Solns

1. (a) False

(b) True

(c) False (Ex: $f(x) = (x-3)(x-2)$, $g(x) = x-2$, $a=2$; $h(2)$ is undefined.)

(d) False (Ex: $f(x) = \frac{1}{x}$, $g(x) = x-2$, $a=2$; $h(2)$ is undefined.)

(e) False

(f) True

(g) False

(h) True

(i) True

(j) True

2. (a) $f(x) = \frac{\sin x}{x}$, $f(x) = \frac{x}{x}$, $f(x) = \frac{x^2+3x}{x}$,

(b) $f(x) = |x|$, $f(x) = \begin{cases} 2x & x \leq 0 \\ 3x & x > 0 \end{cases}$

(c) $f(x) = 2x - 1$

(d) $f(x) = 0$, $f(x) = e^x$

(e) $f(x) = 14$, $f(x) = e^x + 14$

(f) $f(x) = 2^{-x}$, $f(x) = e^{-x}$

(g) $f(x) = \sin(x)$, $f(x) = \cos(x)$

3. (a) $f(x)$ is continuous at $x=a$ if

(1) $\lim_{x\to a} f(x)$ exists,

(2) $f(a)$ exists/is defined, and

(3) $\lim_{x\to a} f(x) = f(a)$

(b) $f(x)$ has a removable discontinuity at $x=a$ if $f(x)$ is not continuous at $x=a$ and $\lim_{x\to a} f(x)$ exists.

(c) The derivative of $f(x)$ at $x=a$ is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

4. (a) Suppose that for all x near, but not equal to, a we have

$$f(x) \le g(x) \le h(x).$$

If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

(b) Suppose that $f(x)$ is continuous on the interval $[a, b]$. If L is a number between $f(a)$ and $f(b)$, then there exists c, $a \le c \le b$, so that $f(c) = L$.

5.(a) Note $f(g(x)) = \begin{cases} (2x)^2 & 2x \neq 0 \\ 4 & 2x = 0 \end{cases}$

$$= \begin{cases} 4x^2 & x \neq 0 \\ 4 & x = 0 \end{cases}$$

$$\lim_{x \to 0} f(g(x)) = \lim_{x \to 0} 4x^2$$

(Remember: $\lim_{x\to 0} f(g(x))$ does not depend on $f(g(0))$, so only the first rule (when $x \neq 0$) is relevant.)

$$= 4 \cdot 0^2$$

$$= \boxed{0}$$

(b) $\lim_{x \to 0} g(x) = \lim_{x \to 0} 2x = 0$

$$f\left(\lim_{x \to 0} g(x)\right) = f(0) = \boxed{4}.$$

(c) $\lim_{x \to 1} \dfrac{x^2 - 1}{x^2 + x - 2} = \lim_{x \to 1} \dfrac{(x+1)(x-1)}{(x+2)(x-1)}$

$$= \lim_{x \to 1} \frac{x+1}{x+2}$$

$$= \boxed{\frac{2}{3}}$$

(d) $$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}}$$

$$= \lim_{x\to 1} \frac{1-\sqrt{x}+\sqrt{x}-\sqrt{x}\cdot\sqrt{x}}{(1-x)(1+\sqrt{x})}$$

$$= \lim_{x\to 1} \frac{1-x}{(1-x)(1+\sqrt{x})}$$

$$= \lim_{x\to 1} \frac{1}{1+\sqrt{x}}$$

$$= \boxed{\frac{1}{2}}$$

(e) $|\sin(4/x)| \leq 1$, so

$$-|x^{1/3}| \leq x^{1/3}\sin(4/x) \leq |x^{1/3}|.$$

Let $f(x) = -|x^{1/3}|$, $g(x) = x^{1/3}\sin(4/x)$, $h(x) = |x^{1/3}|$.

Now $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$, so by the

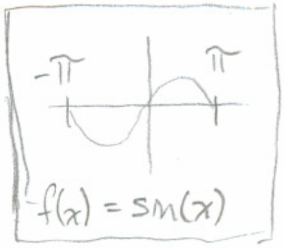
<u>Squeeze</u> <u>Theorem</u>, $\lim_{x\to 0} g(x) = \boxed{0}$.

(f) $x^{1/2} = \sqrt{x}$ is not defined when $x<0$. Therefore

$\lim_{x\to 0^-} x^{1/2}\sin(4/x)$ does not exist,

so $\lim_{x\to 0} x^{1/2}\sin(4/x)$ also $\boxed{\text{does not exist}}$.

(g) $\lim\limits_{x \to \pi^-} \csc(x) = \lim\limits_{x \to \pi^-} \dfrac{1}{\sin(x)}$



As $x \to \pi$ from the left, $\sin(x)$ is small and positive.

Therefore $\dfrac{1}{\sin(x)}$ is large and positive.

So $\lim\limits_{x \to \pi^-} \dfrac{1}{\sin(x)} = \boxed{\infty}$.

(h) $\lim\limits_{x \to -\infty} \dfrac{-x^2 - 4x + 8}{3x^3} \cdot \dfrac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim\limits_{x \to -\infty} \dfrac{-\frac{1}{x} - 4\frac{1}{x^2} + 8\frac{1}{x^3}}{3}$

$$= \frac{\lim\limits_{x \to -\infty} -\frac{1}{x} - \frac{4}{x^2} + 8\frac{1}{x^3}}{\lim\limits_{x \to -\infty} 3}$$

$$= \frac{0 - 0 + 0}{3} = \boxed{0}.$$

(i) $\lim\limits_{x \to -\infty} \dfrac{-x^2 - 4x + 8}{3x^2} \cdot \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim\limits_{x \to -\infty} \dfrac{-1 - 4 \cdot \frac{1}{x} + 8 \cdot \frac{1}{x^2}}{3}$

$$= \frac{\lim\limits_{x \to -\infty} -1 - 4 \cdot \frac{1}{x} + 8 \cdot \frac{1}{x^2}}{\lim\limits_{x \to -\infty} 3}$$

$$= \frac{-1 - 0 + 0}{3} = \boxed{-\frac{1}{3}}.$$

(j) $\lim_{x \to 0^-} \frac{x^2 - 4x + 8}{3x^3}$.

As $x \to 0$ (from the left or the right), $x^2 - 4x + 8 \to 8$

As $x \to 0^-$ (i.e. from the left), $3x^3$ gets small (i.e. close to 0) but negative.

Therefore as $x \to 0^-$, $\frac{x^2-4x+8}{3x^3} \longrightarrow \frac{8}{\text{small negative number}}$.

Therefore $\lim_{x \to 0^-} \frac{x^2-4x+8}{3x^3} = \boxed{-\infty}$.

(k) $\lim_{x \to 0^+} \frac{x^2 - 4x + 8}{3x^2}$.

As $x \to 0$ (from left or right), $x^2 - 4x + 8 \to 8$.

As $x \to 0^+$, $3x^2$ gets small and is positive.

Therefore as $x \to 0^+$, $\frac{x^2-4x+8}{3x^2} \longrightarrow \frac{8}{\text{small positive number}}$.

Therefore $\lim_{x \to 0^+} \frac{x^2-4x+8}{3x^2} = \boxed{\infty}$.

(l) $\lim_{x \to 0} e^{1/x}$.

As $x \to 0^-$, $\frac{1}{x} \to -\infty$. So $\lim_{x \to 0^-} e^{1/x} = 0$.

As $x \to 0^+$, $\frac{1}{x} \to \infty$. So $\lim_{x \to 0^+} e^{1/x} = \infty$.

Because $\lim_{x \to 0^-} e^{1/x} \neq \lim_{x \to 0^+} e^{1/x}$, $\lim_{x \to 0} e^{1/x}$ $\boxed{\text{does not exist}}$.

(m) $\lim_{x\to 0^-} e^{1/x} = \boxed{0}$.

(n) $$\lim_{t\to 0} \frac{t}{\sin(2t)} = \lim_{t\to 0} \frac{1}{2}\cdot\frac{2t}{\sin(2t)}$$

$$= \left(\lim_{t\to 0}\frac{1}{2}\right)\cdot\left(\lim_{t\to 0}\frac{2t}{\sin(2t)}\right)$$

$$= \frac{1}{2}\cdot\left(\lim_{t\to 0}\frac{1}{\frac{\sin(2t)}{2t}}\right)$$

$$= \frac{1}{2}\cdot\frac{\lim_{t\to 0} 1}{\lim_{t\to 0}\frac{\sin(2t)}{2t}}$$

(Note $\lim_{t\to 0}\frac{\sin(t)}{t} = 1$).

$$= \frac{1}{2}\cdot\frac{1}{1} = \boxed{\frac{1}{2}}$$

(o) $$\lim_{x\to 0}\frac{\cos(4x^2)-1}{x^2} = \lim_{x\to 0} 4\left(\frac{\cos(4x^2)-1}{4x^2}\right)$$

$$= 4\cdot\lim_{x\to 0}\frac{\cos(4x^2)-1}{4x^2}$$

$$= 4\cdot 0$$

$$= \boxed{0}.$$

Note: $\lim_{x\to 0}\frac{\cos(x)-1}{x} = 0$

8

6. (a) $f'(x) = \dfrac{\cos(x)\, e^x - e^x(-\sin(x))}{\cos^2(x)}$

$$= \boxed{\dfrac{\cos(x)\, e^x + e^x \sin(x)}{\cos^2(x)}}$$

(b) $f'(x) = 2x\cot(x) + x^2(-\csc^2(x))$

$$= \boxed{2x\cot(x) - x^2\csc^2(x)}$$

(c) $f'(x) = \dfrac{\cos(x^2)\cdot\frac{1}{x} - \ln(x)\left(-\sin(x^2)\cdot 2x\right)}{\cos^2(x^2)}$

$$= \boxed{\dfrac{\cos(x^2)\cdot\frac{1}{x} + \ln(x)\sin(x^2)\cdot 2x}{\cos^2(x^2)}}$$

(d) $f'(x) = \dfrac{d}{dx}\left(\sec\left(e^{5\cos(x^2)}\right)\right)$

$$= \sec\left(e^{5\cos(x^2)}\right)\tan\left(e^{5\cos(x^2)}\right)\cdot\left(\frac{d}{dx}e^{5\cos(x^2)}\right)$$

$$= \sec\left(e^{5\cos(x^2)}\right)\tan\left(e^{5\cos(x^2)}\right)\cdot e^{5\cos(x^2)}\cdot\left(\frac{d}{dx}5\cos(x^2)\right)$$

$$= \sec\left(e^{5\cos(x^2)}\right)\tan\left(e^{5\cos(x^2)}\right) e^{5\cos(x^2)}\cdot 5\,(-\sin(x^2))\left(\frac{d}{dx}x^2\right)$$

$$= \boxed{-10x\sec\left(e^{5\cos(x^2)}\right)\tan\left(e^{5\cos(x^2)}\right) e^{5\cos(x^2)}\sin(x^2)}$$

(e) $f'(x) = \sec^2\left(\sqrt{x^2+1}\right)\cdot\dfrac{d}{dx}\left((x^2+1)^{1/2}\right)$

$$= \sec^2\left(\sqrt{x^2+1}\right)\cdot\frac{1}{2\sqrt{x^2+1}}\cdot 2x$$

$$= \boxed{\sec^2\left(\sqrt{x^2+1}\right)\cdot\frac{x}{\sqrt{x^2+1}}}$$

(f) $f'(x) = \left(\frac{d}{dx} 4x^2\right)\sin(x)\sec(3x) +$

$4x^2 \left(\frac{d}{dx}\sin(x)\right)\sec(3x) +$

$4x^2 \sin(x)\left(\frac{d}{dx}\sec(3x)\right)$

$= 8x\sin(x)\sec(3x) + 4x^2\cos(x)\sec(3x) + 4x^2\sin(x)(\sec(3x)\cdot\tan(3x)\cdot 3)$

$= \boxed{8x\sin(x)\sec(3x) + 4x^2\cos(x)\sec(3x) + 12x^2\sin(x)\sec(3x)\tan(3x)}$

(g) $f'(x) = 4\csc^3(x)\cdot\left(\frac{d}{dx}\csc(x)\right)$

$= 4\csc^3(x)\cdot(-\csc(x)\cdot\cot(x))$

$= \boxed{-4\csc^4(x)\cot(x)}$

(h) $f'(x) = \left(\frac{d}{dx} 3^{\tan(x)}\right)$

$= 3^{\tan(x)}\cdot\ln(3)\cdot\left(\frac{d}{dx}\tan(x)\right)$

$= \boxed{\ln(3)\cdot 3^{\tan(x)}\cdot\sec^2(x)}$

(i) $f(x) = (\cos(x))^{\sin(x)}$ Use Logarithmic differentiation.

$\ln(f(x)) = \sin(x)\cdot\ln(\cos(x))$

$$\frac{d}{dx}\left(\ln(f(x))\right) = \frac{d}{dx}\left(\sin(x)\cdot\ln(\cos(x))\right)$$

$$\frac{1}{f(x)}\cdot f'(x) = \cos(x)\cdot\ln(\cos(x)) + \sin(x)\cdot\frac{1}{\cos(x)}\cdot\left(\frac{d}{dx}\cos(x)\right)$$

$$= \cos(x)\cdot\ln(\cos(x)) + \tan(x)\cdot(-\sin(x))$$

$$= \cos(x)\ln(\cos(x)) - \tan(x)\sin(x).$$

Multiply both sides by $f(x)$:

$$f'(x) = f(x)\left[\cos(x)\ln(\cos(x)) - \tan(x)\sin(x)\right]$$

$$= \boxed{(\cos(x))^{\sin(x)}\left[\cos(x)\ln(\cos(x)) - \tan(x)\sin(x)\right]}$$

7. (a) $f'(x) = 6x^2 - 8x + 5$

$$\boxed{f''(x) = 12x - 8}$$

(b) $f'(x) = e^{\sin(x)}\cdot\cos(x)$

$$f''(x) = \left(\frac{d}{dx}e^{\sin(x)}\right)\cdot\cos(x) + e^{\sin(x)}\cdot\left(\frac{d}{dx}\cos(x)\right)$$

$$= \left(e^{\sin(x)}\cdot\cos(x)\right)\cdot\cos(x) + e^{\sin(x)}(-\sin(x))$$

$$= \boxed{e^{\sin(x)}\cos^2(x) - e^{\sin(x)}\sin(x)}$$

$\Longrightarrow$ Good Luck on Monday!! $\Longleftarrow$