Math221: Chapter 2.5 Selected Exercises

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1 2.5 #8

Find the derivative of $h(x) = (x^3+2)\sqrt{x^5}$.

Solution:

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(\left[x^3+2\right]\left[\sqrt{x^5}\right]\right) \\ &= \left(\frac{d}{dx}(x^3+2)\right)\sqrt{x^5} + (x^3+2)\left(\frac{d}{dx}\sqrt{x^5}\right) \\ &= (3x^2)\sqrt{x^5} + (x^3+2)\left(\frac{d}{dx}x^{5/2}\right) \\ &= 3x^2 \cdot x^{5/2} + (x^3+2)\left(\frac{5}{2}x^{3/2}\right) \\ &= 3x^{9/4} + (x^3+2)\left(\frac{5}{2}x^{3/2}\right) \end{aligned}$$

2 2.5 #20

Find the derivative of $h(x) = \sqrt{(x^2+1)\left(\sqrt{x}+1\right)^3}$.

Solution: first, let us note that $h(x)$ is the composition $h(x) = f(g(x))$ of two simpler functions: the outer function $f(u) = \sqrt{u} = u^{1/2}$ and the inner function $g(x) = (x^2+1)(\sqrt{x}+1)^3$. Therefore, we will want to apply the chain rule. To do so, we first need to compute the derivatives of $f(u)$ and $g(x)$. We note that $f'(u) = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}$. We compute $g'(x)$ as follows, applying the product rule first:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left((x^2+1)\left(\sqrt{x}+1\right)^3\right) \\ &= \left(\frac{d}{dx}(x^2+1)\right)\left(\sqrt{x}+1\right)^3 + (x^2+1)\frac{d}{dx}\left(\left(\sqrt{x}+1\right)^3\right) \\ &= (2x)\left(\sqrt{x}+1\right)^3 + (x^2+1)\left(3\left(\sqrt{x}+1\right)^2 \cdot \frac{d}{dx}\left(\sqrt{x}+1\right)\right) \\ &= 2x\left(\sqrt{x}+1\right)^3 + (x^2+1)\left(3\left(\sqrt{x}+1\right)^2 \cdot \frac{1}{2}x^{-1/2}\right) \\ &= 2x\left(\sqrt{x}+1\right)^3 + \frac{3}{2\sqrt{x}}(x^2+1)\left(\sqrt{x}+1\right)^2. \end{aligned}$$

(To evaluate $\frac{d}{dx}\left(\left(\sqrt{x}+1\right)^3\right)$, we need the chain rule again; this time, the outer function is u^3 and the inner function is $\sqrt{x}+1$.) Finally, we are ready to compute $h'(x)$:

$$\begin{aligned}
h'(x) &= \frac{d}{dx}\left(f\left(g(x)\right)\right) \\
&= f'\left(g(x)\right)\cdot g'(x) \\
&= \frac{1}{2\sqrt{g(x)}}\left(2x\left(\sqrt{x}+1\right)^3+\frac{3}{2\sqrt{x}}\left(x^2+1\right)\left(\sqrt{x}+1\right)^2\right) \\
&= \frac{1}{2\sqrt{\left(x^2+1\right)\left(\sqrt{x}+1\right)^3}}\left(2x\left(\sqrt{x}+1\right)^3+\frac{3}{2\sqrt{x}}\left(x^2+1\right)\left(\sqrt{x}+1\right)^2\right) \\
&= \frac{2x\left(\sqrt{x}+1\right)^3+\frac{3}{2\sqrt{x}}\left(x^2+1\right)\left(\sqrt{x}+1\right)^2}{2\sqrt{\left(x^2+1\right)\left(\sqrt{x}+1\right)^3}}.
\end{aligned}$$

3 2.5 #24

Find an equation of the tangent line to $y=h(x)$ at $x=a$, with $h(x)=\frac{6}{x^2+4}$ and $a=-2$.

Solution: we know the tangent line at $x=a$ has the equation $y=h'(a)\left(x-a\right)+h(a)$, so we must solve for $h'(a)$ and $h(a)$. We compute $h(a)$ by substitution: $h(a)=h(-2)=\frac{6}{(-2)^2+4}=\frac{6}{8}=\frac{3}{4}$. We compute $h'(x)$ by first writing $h(x)$ in a different form

$$h(x)=\frac{6}{x^2+4}=6\left(x^2+4\right)^{-1}$$

and recognizing that $h(x)$ is the composition $h(x)=f\left(g(x)\right)$ of an outer function $f(u)=6u^{-1}$ and an inner function $g(x)=x^2+4$. We compute $f'(u)=-6u^{-2}$ using the power rule and $g'(x)=2x$ using the sum rule and the power rule. Now, we are able to compute

$$\begin{aligned}
h'(x) &= \frac{d}{dx}\left(f\left(g(x)\right)\right) \\
&= f'\left(g(x)\right)\cdot g'(x) \\
&= -6\left(g(x)\right)^{-2}\cdot(2x) \\
&= \frac{-6}{\left(x^2+4\right)^2}\cdot 2x \\
&= \frac{-12x}{\left(x^2+4\right)^2}
\end{aligned}$$

so that

$$\begin{aligned}
h'(a)=h'(-2) &= \frac{-12\cdot(-2)}{\left((-2)^2+4\right)^2} \\
&= \frac{24}{(4+4)^2} \\
&= \frac{24}{64} \\
&= \frac{3}{8}.
\end{aligned}$$

Putting all the pieces together, the equation of the tangent line at $x = -2$ is

$$\begin{aligned} y &= h'(a)(x-a) + h(a) \\ &= \frac{3}{8}(x - (-2)) + \frac{3}{4} \\ &= \frac{3}{8}(x+2) + \frac{3}{4}. \end{aligned}$$

4 2.5 #35

Assume $f(x) = x^3 + 4x - 1$ has an inverse $g(x)$. Find $g'(-1)$.

Solution: because $g(x)$ is an inverse for $f(x)$, we know that

$$g'(x) = \frac{1}{f'(g(x))}$$

(like our other rules for differentiation, you should memorize this formula). To use the formula, we must compute $f'(x) = 3x^2 + 4$. By substitution,

$$\begin{aligned} g'(-1) &= \frac{1}{f'(g(-1))} \\ &= \frac{1}{3(g(-1))^2 + 4}. \end{aligned}$$

To finish solving the problem, we need to know $g(-1)$. Suppose that we are able to find a number z such that $f(z) = -1$. Well, because g is an inverse for f, we have

$$g(f(z)) = z$$

or

$$g(-1) = z.$$

Therefore, to determine the value of $g(-1)$, we must search for a number z with the property that $f(z) = -1$. This can be a tricky thing to do, but the task won't be too difficult for the problems that we might ask. For example, notice that $z = 0$ works because $f(0) = -1$. Therefore $g(-1) = 0$.

Finally, we are able to complete the problem:

$$\begin{aligned} g'(-1) &= \frac{1}{3(g(-1))^2 + 4} \\ &= \frac{1}{3(0)^2 + 4} \\ &= \frac{1}{4}. \end{aligned}$$

5 2.5 #36

Assume $f(x) = x^3 + 2x + 1$ has an inverse $g(x)$. Find $g'(-2)$.

Solution: first, we compute $f'(x) = 3x^2 + 2$. Next, using our formula, we have

$$\begin{aligned} g'(-2) &= \frac{1}{f'(g(-2))} \\ &= \frac{1}{3(g(-2))^2 + 2}. \end{aligned}$$

What is $g(-2)$? Again, we must search for a number z with the property that $f(z) = -2$. Notice that for any positive number z, $f(z)$ will be positive because none of the terms in $f(x)$ involve subtraction. So, we should try values of z which are negative. In fact, $z = -1$ does the trick because

$$f(-1) = (-1)^3 + 2(-1) + 1 = -1 - 2 + 1 = -2.$$

Therefore $g(-2) = -1$ and we are ready to complete the problem:

$$\begin{aligned} g'(-2) &= \frac{1}{3\,(g(-2))^2 + 2} \\ &= \frac{1}{3\,(-1)^2 + 2} \\ &= \frac{1}{3+2} \\ &= \frac{1}{5}. \end{aligned}$$

6 2.5 #40

Assume $f(x) = \sqrt{x^5 + 4x^3 + 3x + 1}$ has an inverse $g(x)$. Find $g'(3)$.

Solution: first, we must compute $f'(x)$. To do so, we rewrite $f(x) = \left(x^5 + 4x^3 + 3x + 1\right)^{1/2}$ and then use the chain rule to compute

$$\begin{aligned} f'(x) &= \frac{1}{2}\left(x^5 + 4x^3 + 3x + 1\right)^{-1/2} \cdot \frac{d}{dx}\left(x^5 + 4x^3 + 3x + 1\right) \\ &= \frac{1}{2}\left(x^5 + 4x^3 + 3x + 1\right)^{-1/2} \cdot \left(5x^4 + 12x^2 + 3\right) \\ &= \frac{1}{2\sqrt{x^5 + 4x^3 + 3x + 1}} \cdot \left(5x^4 + 12x^2 + 3\right) \\ &= \frac{5x^4 + 12x^2 + 3}{2\sqrt{x^5 + 4x^3 + 3x + 1}}. \end{aligned}$$

Next, we use our formula for the derivative of an inverse function:

$$g'(3) = \frac{1}{f'\left(g(3)\right)}.$$

But what is $g(3)$? Once again, we must find a number z so that $f(z) = 3$. After some experimentation, we might try $z = 1$ and discover

$$\begin{aligned} f(1) &= \sqrt{1^5 + 4 \cdot 1^3 + 3 \cdot 1 + 1} \\ &= \sqrt{1 + 4 + 3 + 1} \\ &= \sqrt{9} \\ &= 3. \end{aligned}$$

Therefore $g(3) = 1$. Now, we are ready to finish the problem:

$$\begin{aligned} g'(3) &= \frac{1}{f'(g(3))} \\ &= \frac{1}{f'(1)} \\ &= \frac{2\sqrt{1^5 + 4 \cdot 1^3 + 3 \cdot 1 + 1}}{5 \cdot 1^4 + 12 \cdot 1^2 + 3} \\ &= \frac{2\sqrt{9}}{20} \\ &= \frac{6}{20} \\ &= \frac{3}{10}. \end{aligned}$$