def A graph H is a <u>subgraph</u> of a graph G, denoted $\underline{H \subseteq G}$, if there is an injection $f: V(H) \to V(G)$ such that

(1) $uv \in E(H) \implies f(u)f(v) \in E(G)$.

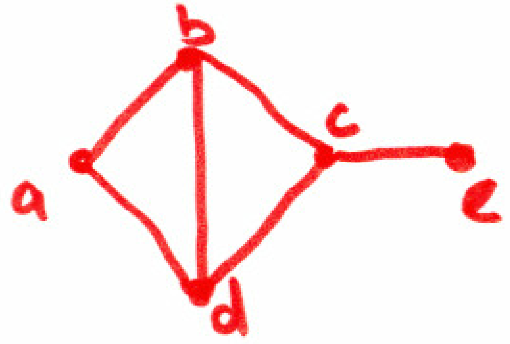
Moreover, if it is the case that

(2) $uv \in E(H) \iff f(u)f(v) \in E(G)$

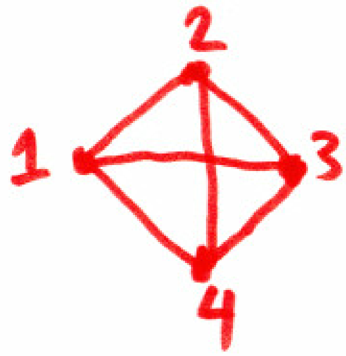
then H is an <u>induced subgraph</u> of G.

If f is a bijection and (2) holds, then f is an <u>isomorphism</u>, we say that H and G are <u>isomorphic</u>, denoted $\underline{H \cong G}$ or sometimes $\underline{H = G}$.

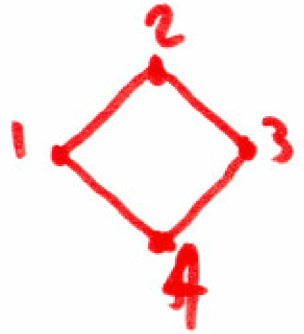
Ex:



G

- 

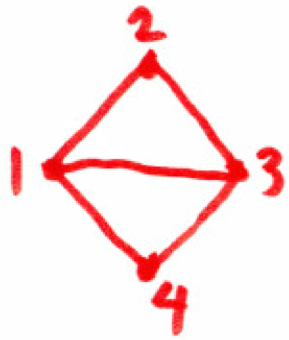
is <u>not</u> a subgraph of G

- 

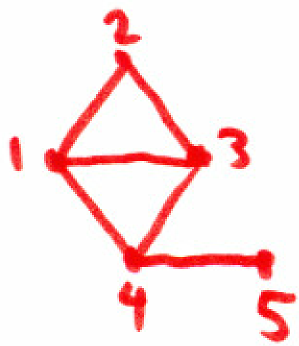
is a subgraph of G (via

v	1	2	3	4
f(v)	a	b	c	d

) but is <u>not</u> an induced subgraph

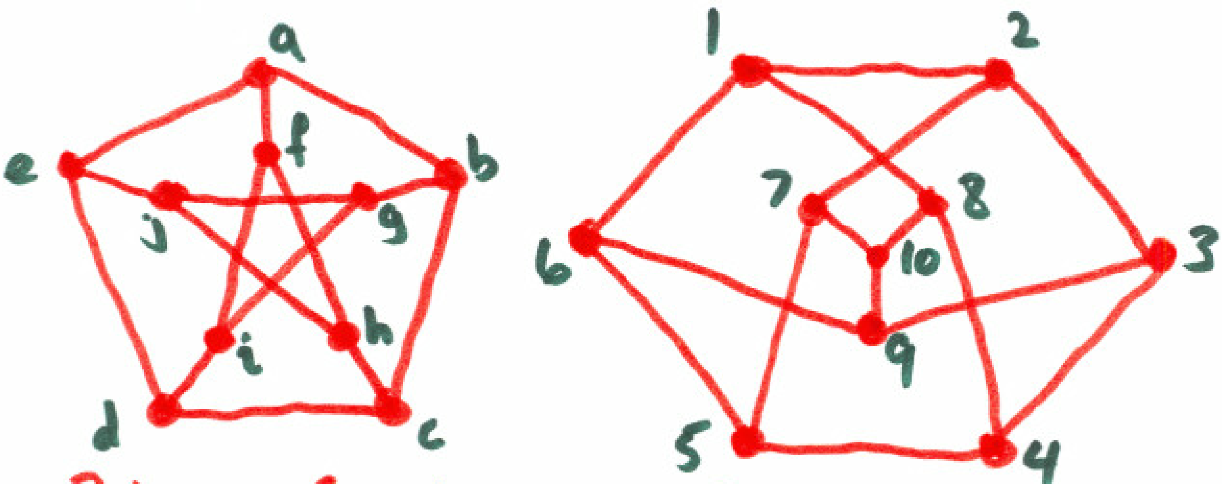
- 

is an induced subgraph of G but is <u>not</u> isomorphic with G.

- 

is isomorphic with G.

Note: it can be tricky to decide whether two given graphs are isomorphic.



Petersen Graph
G

Petersen Graph?
H

In fact, "Graph Isomorphism" is a famous problem and ~~the~~ a topic of ongoing research.

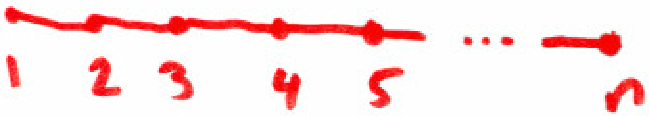
<u>Answer</u>: Yes

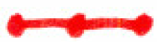
$G \cong H$

G	H
a	9
b	6
c	1
d	2
e	3
f	10
g	5
h	8
i	7
j	4

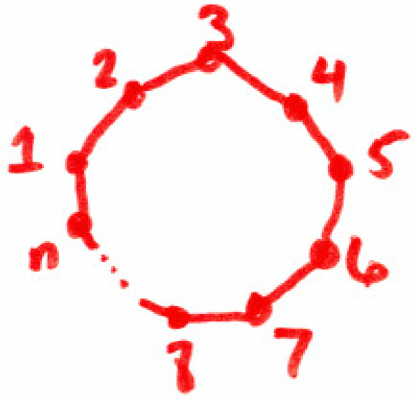
Special Graphs

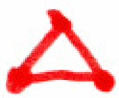
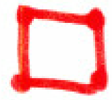
- **Paths**, denoted P_n:



(Ex: $P_1 =$ , $P_2 =$ , $P_3 =$ )

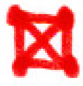
- **Cycles**, denoted C_n; pronounced "n-cycle"



- (Ex: $C_3 =$  $=$ The triangle, $C_4 =$ )

$\Longrightarrow$ The smallest cycle is C_3. $\Longleftarrow$

- **Cliques**, denoted K_n
  - $V(K_n) = \{1, 2, \ldots, n\} = [n]$
  - $E(K_n) = \{A \subseteq [n] \mid |A| = 2\} =$ all edges

(Ex: $K_3 = C_3$, $K_4 =$ )

def A graph G is **acyclic** if it does not contain a cycle ($\{C_3, C_4, C_5, \dots\}$) as a subgraph.

Lemma If G is acyclic, then G contains a vertex of degree at most 1.

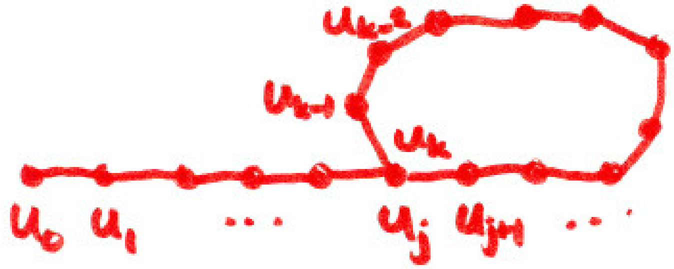
Pf: Suppose for a contradiction that each vertex in G has degree ≥ 2. Starting from an arbitrary vertex u_0, we may find a walk W of length $n = |V(G)|$

$$W = u_0 u_1 u_2 \cdots u_n$$

such that no two consecutive edges traversed by W are the same. Now W is a list of $n+1$ vertices, so the pigeonhole principle implies there are repetitions. Let k be the least integer such that u_k has already appeared in W.

By minimality of k, there is a unique

$0 \le j < k$ such that $u_j = u_k$:



Also by minimality of k, the walk

$$W' = u_j u_{j+1} \cdots u_{k-1} u_k$$

has no repeated vertices, except $u_j = u_k$.

Also, W' has length ≥ 3:

- $j < k$, so length of W is ≥ 1

- $u_j u_{j+1} \in E(G)$, so $u_{j+1} \neq u_j$ (length ≥ 2)

- No two consecutive edges in W (and hence W') are the same; so $u_j u_{j+1} \neq u_{j+1} u_{j+2}$ requires $u_{j+2} \neq u_j$
- Also, $u_{j+1} u_{j+2} \in E(G)$, so $u_{j+1} \neq u_{j+2}$.

Therefore u_j, u_{j+1}, u_{j+2} are all distinct and so the length of W' is at least 3.

Hence, W' is a cycle in G. ∎

Thm If G is an n-vertex acyclic graph, then G has at most $n-1$ edges.

Pf: By induction on n.

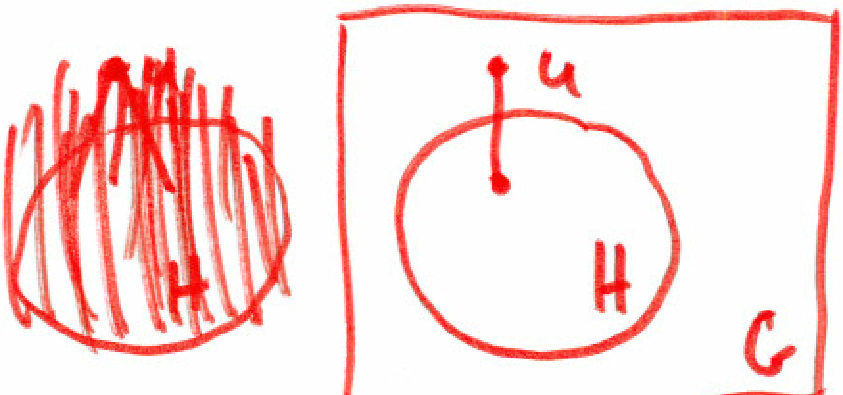
If $n=1$, then G has 0 edges.

If $n \geq 2$, then by our lemma, G has a vertex u of degree $d(u) \leq 1$.

Let $H = G - u$; that is

$$V(H) = V(G) - \{u\}$$
$$E(H) = E(G) - \{e \in E(G) \mid u \text{ is an endpoint of } e\}$$

In picture,



Now H is an $(n-1)$-vertex graph with $|E(G)| - d(u)$ edges. By the inductive hypothesis, $|E(G)| - d(u) \leq (n-1) - 1 = n-2$. Therefore

$|E(G)| \le n-2+d(u) \le n-1$, so G has at most $n-1$ edges. ■

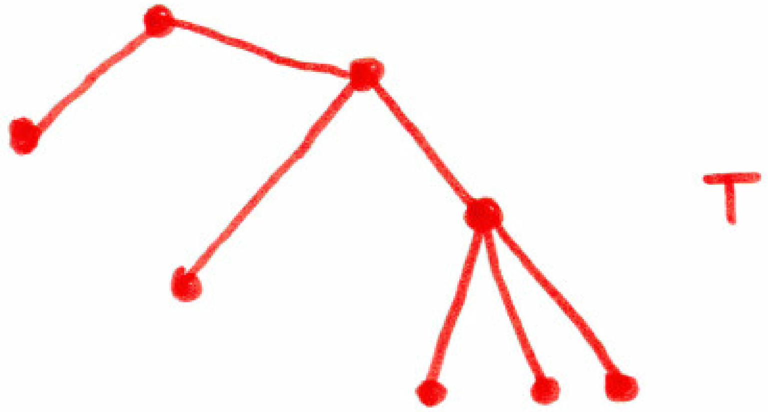
def A <u>tree</u> is a connected, acyclic graph. A <u>rooted tree</u> is a tree T and a distinguished vertex $r \in V(T)$ called the ~~root~~ <u>root</u> of T. A <u>forest</u> is an acyclic graph.

<u>Thm</u> If T is an n-vertex tree, then T has exactly $n-1$ edges.

<u>Pf</u>: Because T is connected, T has at least $n-1$ edges. Because T is acyclic, T has at most $n-1$ edges. ■

<u>Note</u>: If G is a forest, each component of G is a tree. (The converse also holds: if each component of G is a tree, then G is a forest.)

Ex:



- T has 8 vertices and 7 edges.

Facts:

- T is a tree iff (if and only if) $\forall\, u,v \in V(T)$ there is a unique uv-path in T.
- Every tree on at least 2 vertices has at least 2 vertices with degree 1; these vertices are called leaves.