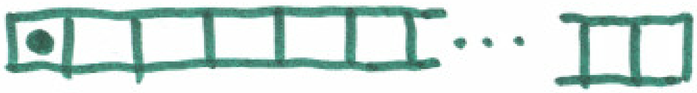
A Random Walk

- Suppose we have a line of n spaces and a game piece starts in the left-most spot:



- The game piece moves as follows. Until the game piece falls off the board, we flip a fair coin and move the token <u>left</u> with probability $\frac{1}{2}$ and <u>right</u> with probability $\frac{1}{2}$.

- We win the game if the token falls off the far right edge.

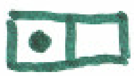
- What is the probability that we win?

<u>Solution Strategy</u>: let p_n be the probability that we win the game when our board contains n spaces.

We will find a recurrence for p_n and solve the recurrence.

<u>Ex</u>

-  $p_1 = \frac{1}{2}$

-  $p_2 = Pr(\text{win} \mid \text{first move left}) \cdot Pr(\text{1st left})$
$+ Pr(\text{win} \mid \text{first move right}) \cdot Pr(\text{1st right})$

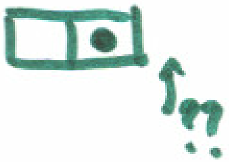
$$= 0 \cdot \tfrac{1}{2} + (1 - p_2) \cdot \tfrac{1}{2}$$

$$= \tfrac{1}{2} - \tfrac{1}{2} p_2$$

So $\frac{3}{2} p_2 = \frac{1}{2}$ and $p_2 = \frac{1}{3}$.

<u>Soln</u>: We use "win" to denote the event that the token falls off the right edge.

We conditionalize based upon the first move, <u>left</u> or <u>right</u>.



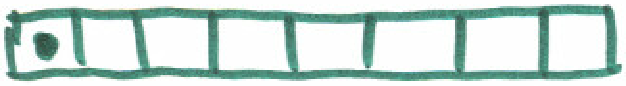
Pr(win | first move right) = Pr(win starting from 2nd square)

= Pr(~~we~~ don't win starting from 1st square)

$= Pr(\overline{win})$

$= 1 - Pr(win)$

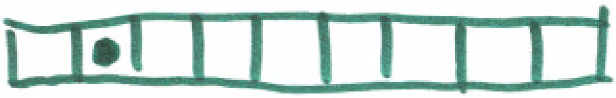
$= 1 - p_2$



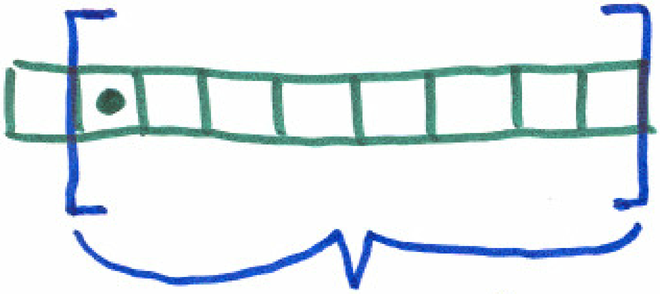
$$p_n = Pr(\text{win}) = Pr(\text{win} \mid \text{1st left}) \cdot Pr(\text{1st left}) + Pr(\text{win} \mid \text{1st right}) \cdot Pr(\text{1st right})$$

$$= 0 \cdot \tfrac{1}{2} + Pr(\text{win} \mid \text{1st right}) \cdot \tfrac{1}{2}$$

- Therefore, we need to compute $Pr(\text{win} \mid \text{1st right})$. Note that $Pr(\text{win} \mid \text{1st right})$ is ~~the same~~ just ~~as~~ the probability that our token falls off the right side, starting from the second square.



- Let A be the event that our token falls off to the right, starting from the second square. Note: $Pr(A) = Pr(\text{win} \mid \text{1st right})$
- ~~Note that~~ Consider two cases, depending upon how the token leaves the smaller board for the first time:



smaller board with $n-1$ spaces

- Let B be the event that the token falls off the smaller board to the right.

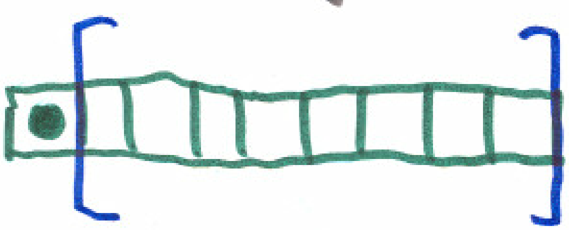
- Conditionalize on B:

$$\Pr(A) = \Pr(A \mid B)\cdot \Pr(B) + \Pr(A \mid \overline{B})\cdot \Pr(\overline{B})$$

$$= \quad 1 \quad \cdot\, p_{n-1} + \Pr(A \mid \overline{B})\cdot (1 - p_{n-1})$$

- What is $\Pr(A \mid \overline{B})$? Well, if we are told that the token falls off the smaller board to the left (i.e. $\overline{B}$ occurs)* then at some point, the token is again on the leftmost square:

* See Lecture for a slight correction



From this position, the probability that

4.1

Let C be the event that the token falls off the smaller board to the left

Let D be the event that the token never falls of the smaller board.

$$\Pr(A) = \Pr(A|B)\cdot\Pr(B) + \Pr(A|C)\cdot\Pr(C) + \Pr(A|D)\cdot\underbrace{\Pr(D)}_{=0}$$

$$= \Pr(A|B)\cdot\Pr(B) + \Pr(A|C)\cdot\Pr(C)$$

$$= 1 \cdot p_{n-1} + \Pr(A|C)\cdot(1-p_{n-1})$$

$$= p_{n-1} + p_n\cdot(1-p_{n-1})$$

the token falls off to the right is p_n.

Therefore $\Pr(A \mid \overline{B}) = p_n$.

- Hence,

$$\Pr(A) = p_{n-1} + p_n(1 - p_{n-1})$$

and

$$p_n = \frac{1}{2}\left[p_{n-1} + p_n(1 - p_{n-1})\right]$$

$$= \frac{1}{2}p_{n-1} + \frac{1}{2}p_n(1 - p_{n-1})$$

so solving for p_n, we have:

$$p_n\left(1 - \frac{1}{2}(1 - p_{n-1})\right) = \frac{1}{2}p_{n-1}$$

$$p_n\left(\frac{1}{2}\left[1 + p_{n-1}\right]\right) = \frac{1}{2}p_{n-1}$$

$$\boxed{p_n = \frac{p_{n-1}}{1 + p_{n-1}}}, \quad p_1 = \frac{1}{2}$$

- At first, this recurrence looks intimidating; none of our usual methods seem to apply.

- Don't forget about <u>guess</u> and <u>check</u>!

n	1	2	3	4	...
P_n	$1/2$	$1/3$	$1/4$	$1/5$	

- $P_2 = \dfrac{P_1}{1+P_1} = \dfrac{1/2}{1+\frac{1}{2}} = \dfrac{\frac{1}{2}}{3/2} = \dfrac{1}{3}$

- $P_3 = \dfrac{P_2}{1+P_2} = \dfrac{1/3}{1+\frac{1}{3}} = \dfrac{\frac{1}{3}}{4/3} = \dfrac{1}{4}$

- $P_4 = \dfrac{P_3}{1+P_3} = \dfrac{\frac{1}{4}}{1+\frac{1}{4}} = \dfrac{\frac{1}{4}}{5/4} = \dfrac{1}{5}$

- <u>Guess</u> $P_n = \dfrac{1}{n+1}$.

<u>Check</u>: By induction on $n \geq 1$. If $n=1$, the formula is correct. If $n \geq 2$, then

$$P_n = \frac{P_{n-1}}{1+P_{n-1}}$$

(by I.H.) $= \dfrac{\frac{1}{n}}{1+\frac{1}{n}}$

$$= \frac{\frac{1}{n}}{\frac{n+1}{n}}$$

$$= \frac{1}{n+1} \quad \checkmark$$

so our formula is correct. We conclude

$$\boxed{P_n = \frac{1}{n+1}}$$

- Note that as the board gets longer, $P_n \to 0$, which makes sense.

- We can use our formula $p_n = \frac{1}{n+1}$ to compute the expected number of moves needed before the token falls off (either end) and the game ends.

- ~~Let W be the number of moves required for the token to fall off the board.~~

  Let $T(\cdot)$ be the function whose value $T(n)$ is the expected number of moves required for the token to fall off a board of length n.

  ~~Note: $T(n) = E[W]$.~~

- Now, for a fixed n, let W be the number of moves required for the token to fall off the board of length n.

  Note: $T(n) = E[W]$.

- How do we compute $E[W]$? Conditionalize on the first move (<u>left</u> or <u>right</u>):

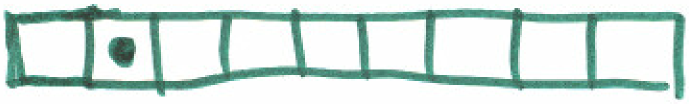
$$E[W] = E[W \mid \text{1st left}] \cdot \Pr(\text{1st left}) + E[W \mid \text{1st right}] \cdot \Pr(\text{1st right})$$

$$= 1 \cdot \tfrac{1}{2} + E[W \mid \text{1st right}] \cdot \tfrac{1}{2}$$

- What is $E[W \mid \text{1st right}]$? Well, ~~this is just one more than the~~ given that $\omega \in \Omega$ is an outcome whose first move is right, $W(\omega) = 1 + W_0(\omega)$ where $W_0(\omega)$ is the number of moves that ω uses to fall off the board, starting from the second square. By Linearity of Expectation,

$$E[W \mid \text{1st right}] = 1 + E[W_0 \mid \text{first right}]$$

- Let X be the number of moves required to fall off the board starting from the second square:

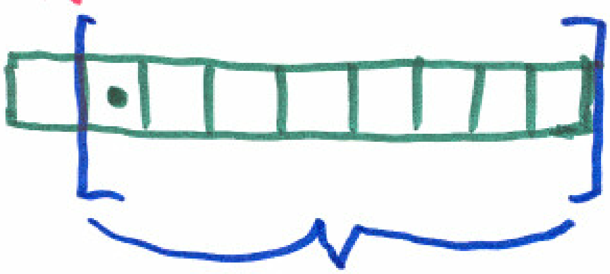


so that $E[X] = E[W_0 \mid \text{first right}]$ and

$E[W \mid \text{1st right}] = 1 + E[X]$.

- To compute X, we define

Y = number of moves until we fall off of the smaller board for first time



Z = number of moves between falling off smaller board for the first time and falling off ~~the~~ the larger board.

Note $X = Y + Z$, so Linearity of Expectation gives us

$$E[X] = E[Y] + E[Z]$$

$$= T(n-1) + E[Z]$$

- How do we compute $E[Z]$? Conditionalize on the event B that the token falls off the smaller board for the first time to the right:

$$E[Z] = E[Z \mid B] \cdot Pr[B] + E[Z \mid \bar{B}] \cdot Pr[\bar{B}]$$

$$= 0 \cdot \frac{1}{n} + E[Z \mid \bar{B}] \cdot \frac{n-1}{n}$$

$$= T(n) \cdot \frac{n-1}{n}$$

(by our formula, $Pr[B] = p_{n-1} = \frac{1}{n}$)

because, given that $\bar{B}$ occurs, then Z starts counting moves from the first time ~~at wh~~ the token falls off the smaller board to the left, placing the token again in the leftmost square.

• Putting it all together,

$$
\begin{aligned}
T(n) &= E[W] \\
&= 1\cdot\tfrac{1}{2} + E[W \mid \text{first right}]\cdot\tfrac{1}{2} \\
&= \tfrac{1}{2}\left(1 + E[W \mid \text{first right}]\right) \\
&= \tfrac{1}{2}\left(2 + E[X]\right) \\
&= \tfrac{1}{2}\left(2 + T(n-1) + \tfrac{n-1}{n}\cdot T(n)\right)
\end{aligned}
$$

and so

$$T(n)\left[1 - \frac{n-1}{2n}\right] = \frac{1}{2}\left(2 + T(n-1)\right)$$

$$T(n)\left[\frac{2n-n+1}{2n}\right] = \frac{1}{2}\left(2 + T(n-1)\right)$$

$$T(n)\left[\frac{n+1}{2n}\right] = \frac{1}{2}\left(2 + T(n-1)\right)$$

$$\boxed{\begin{aligned} T(n) &= \frac{2n}{2(n+1)}\left(2 + T(n-1)\right) \\ &= \frac{n}{n+1}\left(2 + T(n-1)\right) \end{aligned}} \qquad T(1) = 1$$

- Again, this recurrence looks tricky but guess and check comes to the rescue.

$$T(2) = \frac{2}{3}(2+1) = 2$$

$$T(3) = \frac{3}{4}(2+2) = 3$$

$$T(4) = \frac{4}{5}(2+3) = 4$$

- <u>Guess</u>: $T(n) = n$

  <u>Check</u>: If $n=1$, the formula works. If $n \geq 2$,

$$T(n) = \frac{n}{n+1}(2 + T(n-1))$$

(by I.H.) $$= \frac{n}{n+1}(2+n-1)$$

$$= \frac{n}{n+1}(n+1)$$

$$= n \quad ✓$$

- So $\boxed{T(n) = n}$ and this makes sense; the longer the board is, the more moves we must make on average to fall off.