CSTBC Homework 1 Solutions

19th June 2007

1 How Many?

Let $A = \{n \mid 1 \leq n \leq 2007 \text{ and } n \text{ is divisible by 2 or 5}\}$. Compute $|A|$.

(Hint: let

$$B = \{n \mid 1 \leq n \leq 2007 \text{ and } n \text{ is divisible by } 2\}$$

and

$$C = \{n \mid 1 \leq n \leq 2007 \text{ and } n \text{ is divisible by } 5\}.$$

What are $|B|$, $|C|$, and $|B \cap C|$?)

Solution Because every other number is even, there are $2006/2 = 1003$ even numbers in the set $\{1, 2, \ldots, 2006\}$. Because 2007 is odd, there are also 1003 even numbers in the set $\{1, 2, \ldots, 2007\}$. Therefore $|B| = 1003$. Similarly, every fifth number in the set $\{1, 2, \ldots, 2005\}$ is divisible by 5, so that there are $2005/5 = 401$ numbers in $\{1, 2, \ldots, 2005\}$ that are divisible by 5. Because 2006 and 2007 are not divisible by 5, we conclude that $|C| = 401$.

Note that an integer n is divisible by 2 and 5 if and only if n is divisible by 10. Therefore $B \cap C$ is the set of all integers $1 \leq n \leq 2007$ that are divisible by 10. Using the same technique as before, we conclude that $|B \cap C| = 200$. Because $A = B \cup C$, we use an identity from Lecture 1 to conclude

$$|A| = |B \cup C| = |B| + |C| - |B \cap C| = 1003 + 401 - 200 = 1204.$$

2 Injective? Surjective?

For each function below, determine whether the function $f : A \to B$ is bijective, injective but not surjective, surjective but not injective, or neither injective nor surjective. In problems 6-8, $n \geq 1$ is an integer.

1. $A = \{0, 1, 2, \ldots\}$, $B = \{0, -1, -2, \ldots\}$, $f(n) = -n$.
2. $A = \{0, 1, 2, \ldots\}$, $B = \{0, 1, 2, \ldots\}$, $f(n) = n + 1$.
3. $A = \{0, 1, 2, \ldots\}$, $B = \{1, 2, 3, \ldots\}$, $f(n) = n + 1$.
4. $A = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $B = \{0, 1, 2, \ldots\}$, $f(n) = n^2$.
5. $A = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $B = \{0, 1, 2, \ldots\}$, $f(n) = |n|$. (Recall that for a real number x, we denote the absolute value of x by $|x|$. That is, if $x \geq 0$, then $|x| = x$ and $|x| = -x$ otherwise.)
6. $\mathcal{U} = \{1, 2, \ldots, n\}$, $A = B = \mathcal{P}(\mathcal{U})$, $f(S) = \overline{S}$.
7. $\mathcal{U} = \{1, 2, \ldots, n\}$, $A = B = \mathcal{P}(\mathcal{U})$, $f(S) = S \cup \{1\}$.
8. $\mathcal{U} = \{1, 2, \ldots, n\}$, $A = B = \mathcal{P}(\mathcal{U})$,

$$f(S) = \begin{cases} S \cup \{1\} & 1 \notin S \\ S - \{1\} & 1 \in S \end{cases}.$$

Solution

1. bijective
2. injective but not surjective (nothing maps to $0 \in B$)
3. bijective
4. neither
5. surjective but not injective
6. bijective
7. neither
8. bijective

3 An Injection

Let $n \geq 1$ be an integer, let $\mathcal{U} = \{1, 2, \ldots, n\}$, and let $\mathcal{A} = \{A \subseteq \mathcal{U} \mid |A| = k\}$; that is, $\mathcal{A}$ consists of all the sets $A \subseteq \mathcal{U}$ which have size k. Construct an injection $f : \mathcal{A} \to \mathcal{U}^k$. What can we conclude about $|\mathcal{A}|$?

Solution Consider an element $A \in \mathcal{A}$. We must choose an element $f(A) \in \{1, 2, \ldots, n\}^k$ in such a way that f is injective. Because $A \in \mathcal{A}$, A contains k elements; define $a_1, \ldots, a_k$ so that $A = \{a_1, a_2, \ldots, a_k\}$. We set $f(A) = (a_1, a_2, \ldots, a_k)$. Let us check that f is an injective function. If $A_1 \neq A_2$, then there is an element $j \in \mathcal{U}$ such that either $j \in A_1 - A_2$ or $j \in A_2 - A_1$. Hence j will appear in exactly one of the k-tuples $f(A_1)$ and $f(A_2)$, and so $f(A_1) \neq f(A_2)$.

Because $f : \mathcal{A} \to \mathcal{U}^k$ is an injection, we conclude that $|\mathcal{A}| \leq |\mathcal{U}^k| = n^k$.

4 Pairwise Disjoint Families

Let $n \geq 1$ be an integer and let $\mathcal{U} = \{1, 2, \ldots, n\}$. We say that a family $\mathcal{A} \subseteq \mathcal{P}(\mathcal{U})$ of sets is pairwise disjoint if, for each pair of sets A, $B \in \mathcal{A}$, we have that A and B are disjoint (that is, $A \cap B = \emptyset$).

1. Prove that if $\mathcal{A} \subseteq \mathcal{P}(\mathcal{U})$ is a pairwise disjoint family of sets, then $|\mathcal{A}| \leq n + 1$.
2. Find a pairwise disjoint family $\mathcal{A} \subseteq \mathcal{P}(\mathcal{U})$ with $|\mathcal{A}| = n + 1$.
3. Besides the family $\mathcal{A}$ that you found in part (2), are there any other pairwise disjoint families $\mathcal{B} \subseteq \mathcal{P}(\mathcal{U})$ with $|\mathcal{B}| = n + 1$?

Solution

1. Let $\mathcal{A} \subseteq \mathcal{P}(\mathcal{U})$ be a pairwise disjoint family, and let $\mathcal{B} = \mathcal{A} - \{\emptyset\}$. Because we obtain $\mathcal{B}$ from $\mathcal{A}$ by removing at most one of the sets in $\mathcal{A}$, we have that $|\mathcal{B}| \geq |\mathcal{A}| - 1$, or equivalently $|\mathcal{A}| \leq |\mathcal{B}| + 1$. Therefore, it suffices to show that $|\mathcal{B}| \leq n$. Define $B_1, B_2, \ldots, B_k$ so that $\mathcal{B} = \{B_1, B_2, \ldots, B_k\}$; note that $|\mathcal{B}| = k$. Define $B = B_1 \cup B_2 \cup \cdots \cup B_k$. Because $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{A}$ is a pairwise disjoint family, $\mathcal{B}$ is also pairwise disjoint. Hence, B is the disjoint union of the sets $B_1, B_2, \ldots, B_k$ and therefore $|B_1| + |B_2| + \cdots + |B_k| = |B|$. Of course, $\emptyset \notin \mathcal{B}$ and so each set B_j, $1 \leq j \leq k$, has size at least one. Also, because $B \subseteq \mathcal{U}$, $|B| \leq |\mathcal{U}| = n$. It follows that

$$k \leq |B_1| + |B_2| + \cdots + |B_k| = |B| \leq n$$

   and so $|\mathcal{B}| = k \leq n$.

2. Let $\mathcal{A} = \{\emptyset, \{1\}, \{2\}, \ldots, \{n\}\}$; clearly, $\mathcal{A}$ is pairwise disjoint and $|\mathcal{A}| = n + 1$.

3. No, there are no other such families. We give two proofs. Our second proof gives another possible solution to part (1).
   Proof 1: Let $\mathcal{A} \neq \{\emptyset, \{1\}, \{2\}, \ldots, \{n\}\}$ be a pairwise disjoint family. For this special case, we strengthen our our proof in part (1) to conclude that $|\mathcal{A}| \leq n$. First, suppose that $\emptyset \notin \mathcal{A}$. In our proof above, we show that a pairwise disjoint family which does not contain the emptyset has size at most n; hence, if $\emptyset \notin \mathcal{A}$, then $|\mathcal{A}| \leq n$. Otherwise, suppose that $\emptyset \in \mathcal{A}$, write $\mathcal{A} = \{\emptyset, A_1, \ldots, A_k\}$, and let $A = A_1 \cup \cdots \cup A_k$. If for some set A_j we have $|A_j| \geq 2$, then similarly to our proof above, we have $k + 1 \leq |A_1| + \cdots + |A_k| = |A| \leq n$, and so $|\mathcal{A}| = k + 1 \leq n$. Finally, if each of the sets A_j have size at most one, then $\mathcal{A} \neq \{\emptyset, \{1\}, \{2\}, \ldots, \{n\}\}$ forces $|\mathcal{A}| \leq n$.

   Proof 2: Let $\mathcal{A}$ be a pairwise disjoint family of subsets of $\mathcal{U}$ whose size is as large as possible; in other words, let $\mathcal{A}$ be a maximum pairwise disjoint family of subsets of $\mathcal{U}$. Note that if $\mathcal{A}$ is a pairwise disjoint family, $\mathcal{A} \cup \{\emptyset\}$ is also a pairwise disjoint family. Because $\mathcal{A}$ is a largest pairwise disjoint family, it must be that $\emptyset \in \mathcal{A}$. Next, note that if $A \in \mathcal{A}$, then $|A| \leq 1$. Indeed, if $|A| \geq 2$, then we can partition A into two nonempty subsets; that is, we can find sets B and C with $A = B \cup C$ and $B \cap C = \emptyset$. In this case, $(\mathcal{A} - \{A\}) \cup \{B, C\}$ is a pairwise disjoint family that is larger than $\mathcal{A}$, which is impossible. Therefore $\mathcal{A}$ consists of the emptyset plus some of the singleton sets. If $\mathcal{A}$ is missing any of the singleton sets, we could add these sets and find a larger pairwise disjoint family, also impossible. It follows that $\mathcal{A} = \{\emptyset, \{1\}, \{2\}, \ldots, \{n\}\}$.

5 More Pairwise Intersecting Families

Let $n \geq 3$ be an integer and let $\mathcal{U} = \{1, 2, \ldots, n\}$. Recall from lecture 1 that a family $\mathcal{A} \subseteq \mathcal{P}(\mathcal{U})$ of sets is pairwise intersecting if, for each pair of sets $A, B \in \mathcal{A}$, we have that $A \cap B \neq \emptyset$. In lecture 1, we saw that if $\mathcal{A} \subseteq \mathcal{P}(\mathcal{U})$ is pairwise intersecting, then $|\mathcal{A}| \leq 2^{n-1}$. We also found that $\mathcal{A} = \{A \subseteq U \mid 1 \in A\}$ is an example of a pairwise intersecting family of size 2^{n-1}, but this family has the property that there exists an element $j \in \mathcal{U}$ (namely, $j = 1$), such that for each $A \in \mathcal{A}$, $j \in A$.

Construct a pairwise intersecting family $\mathcal{B} \subseteq \mathcal{P}(\mathcal{U})$ of size $|\mathcal{B}| = 2^{n-1}$ which fails to have this property. That is, you are asked to find a family $\mathcal{B} \subseteq \mathcal{P}(\mathcal{U})$ with the following properties:

1. $\mathcal{B}$ is pairwise intersecting,
2. $|\mathcal{B}| = 2^{n-1}$, and
3. for each $j \in \mathcal{U}$, there exists some $B \in \mathcal{B}$ such that $j \notin B$.

(Hint: you may find the proof that $|\mathcal{B}| \leq 2^{n-1}$ helpful.)

Solution As we saw in Lecture 1, complementation groups the subsets of $\mathcal{U}$ into $r = 2^{n-1}$ complementary pairs. Let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be the complementary pairs of $\mathcal{P}(\mathcal{U})$, so that for each $1 \leq j \leq r$, $\mathcal{S}_j = \{B, \overline{B}\}$ for some set $B \subseteq \mathcal{U}$. For each $1 \leq j \leq r$, let B_j be the larger of the two sets in $\mathcal{S}_j$ (if both sets in $\mathcal{S}_j$ have the same size, choose B_j arbitrarily). Note that because $|B| + |\overline{B}| = n$, we have that $|B_j| \geq n/2$. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_r\}$; clearly $|\mathcal{B}| = r = 2^{n-1}$, so (2) is satisfied. We claim that $\mathcal{B}$ is a pairwise intersecting family. Consider $B_i, B_j \in \mathcal{B}$ and suppose for a contradiction that B_i and B_j do not intersect. In this case,

$$|B_i \cup B_j| = |B_i| + |B_j| \geq n/2 + n/2 = n,$$

and so $\mathcal{U} = B_i \cup B_j$. That is, $\{B_i, B_j\}$ is a complementary pair. But $\mathcal{B}$ selects just one set from each of the complementary pairs, so it is impossible that both B_i and B_j are members of $\mathcal{B}$; this is a contradiction. Therefore it must be that B_i and B_j intersect after all. Therefore (1) is satisfied.

It remains to check that (3) is satisfied. In fact, if $n \geq 3$, then $\mathcal{B}$ satisfies (3). Consider $j \in \mathcal{U}$; note that $\mathcal{S} = \left\{ \{j\}, \overline{\{j\}} \right\}$ is a complementary pair. When $n \geq 3$, $\left| \overline{\{j\}} \right| = n - 1 > 1$, and therefore $j \notin \overline{\{j\}} \in \mathcal{B}$. (For $n \in \{1, 2\}$, it is impossible to obtain a family $\mathcal{B}$ which satisfies (1), (2), and (3).)