Ordered star colorings

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West Virginia University

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Problem advertisement

- A k-coloring $\phi : [n] \to [k]$ is good if every 3-AP $a-d, a, a+d$ with $\phi(a-d) = \phi(a+d)$ has $\phi(a) > \phi(a-d)$.

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- Let $f(n)$ be the min. k such that good k-colorings of $[n]$ exist.

Question

What is the growth rate of $f(n)$?

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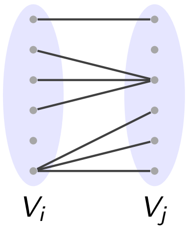
- A star-coloring of G is a partition of $V(G)$ into independent sets $V_1, \ldots, V_k$ such that $V_i \cup V_j$ induces a star forest.

Ordered star colorings



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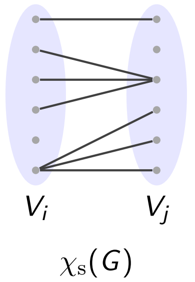

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- The star chromatic number, denoted $\chi_s(G)$, is the minimum number of parts needed.

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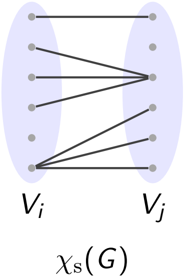

- An ordered star-coloring of G is a partition of $V(G)$ into independent sets $V_1, \ldots, V_k$ such that for $i < j$, the vertices in $V_i \cup V_j$ induce a star forest with centers in V_j.

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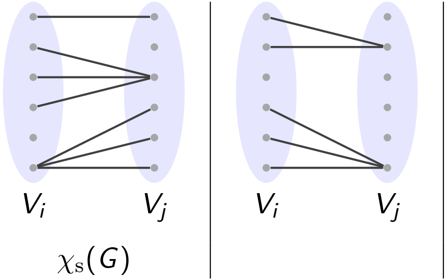

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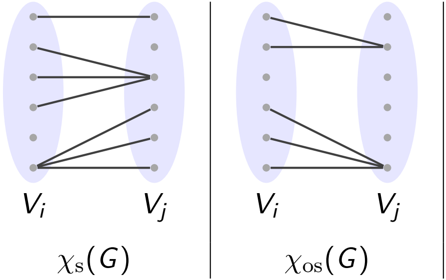

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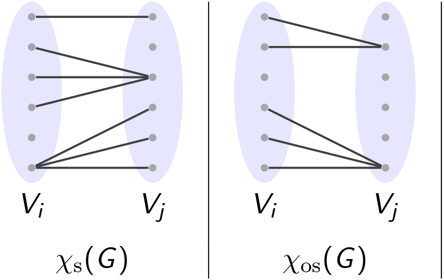

- A distance 2-coloring of G is a partition of $V(G)$ into independent sets $V_1, \ldots, V_k$ such that $V_i \cup V_j$ induces a graph with max. degree at most 1.

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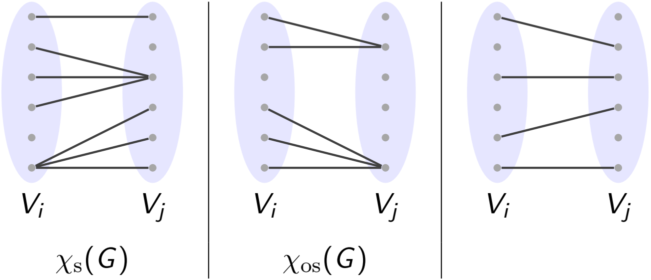

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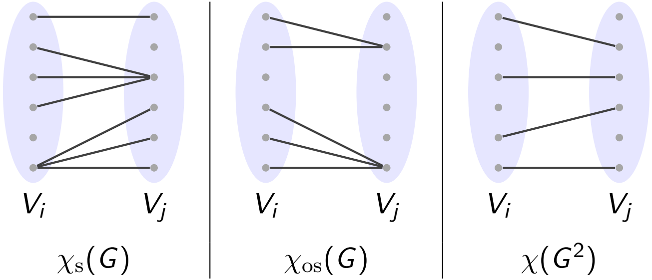

- A distance 2-coloring of G is a partition of $V(G)$ into independent sets $V_1, \dots, V_k$ such that $V_i \cup V_j$ induces a graph with max. degree at most 1.
- It is also $\chi(G^2)$, the chromatic number of the distance square graph G^2.

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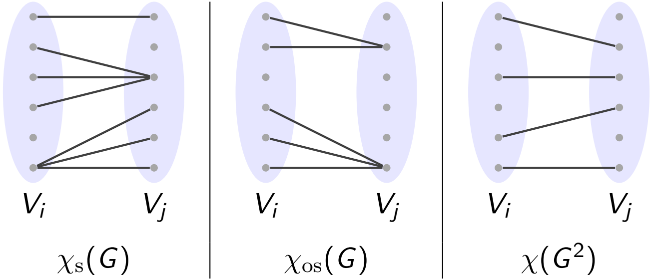

- Clearly, $\chi_s(G) \leq \chi_{os}(G) \leq \chi(G^2)$.

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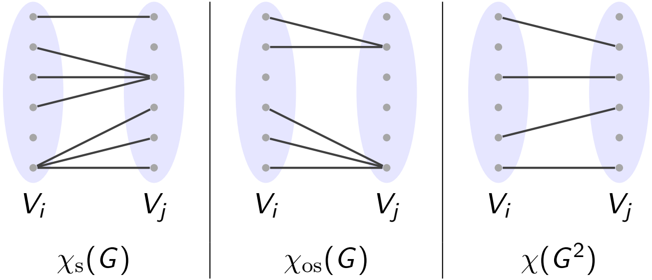
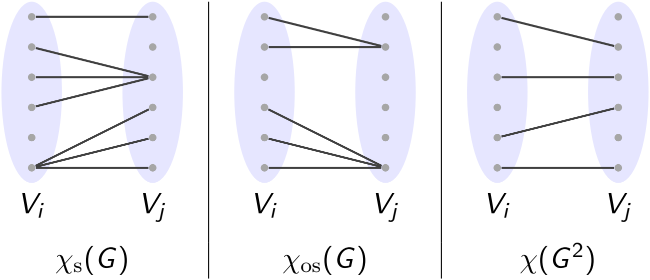


- Clearly, $\chi_s(G) \leq \chi_{os}(G) \leq \chi(G^2)$.
- In an ordered star coloring, if u and v share a color, then every common neighbor has a higher color.

Ordered star colorings



Proposition

$\chi_{\mathrm{os}}(G) \geq \delta(G) + 1$.

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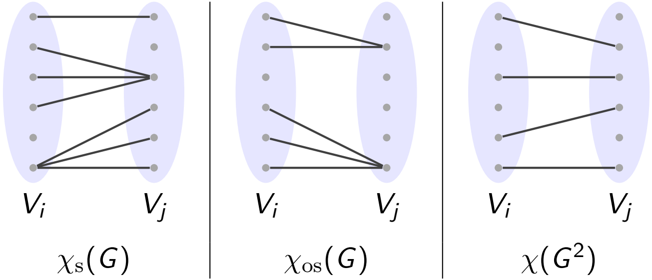

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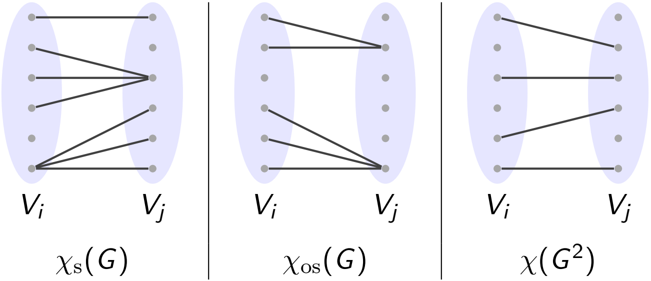

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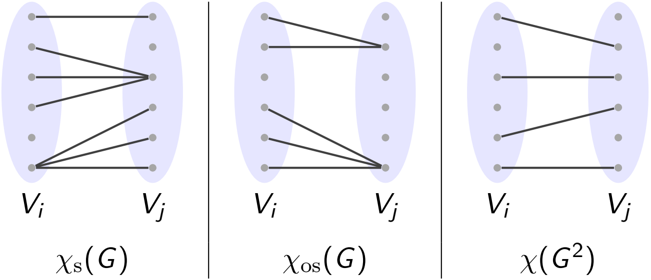

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- Let $v \in V(G)$ have smallest color.
- All colors in $N(v)$ are distinct.
- With v, this gives $d(v) + 1$ colors.

□

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Theorem (KNS 2015)

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- *There is an n-vertex 2-degenerate graph G such that* $\chi_{\mathrm{os}}(G) \geq \Omega(n^{1/3})$.

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- The *d*-dimensional cube, denoted Q_d has vertex set $\{0, 1\}^d$ with two vertices adjacent if and only if they differ in exactly one coordinate.

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- Trickier for general d.

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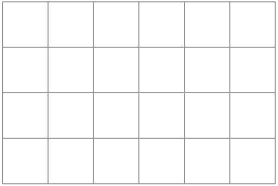

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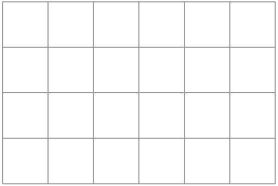

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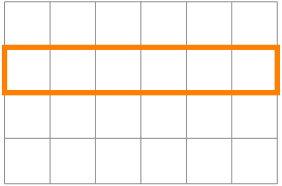

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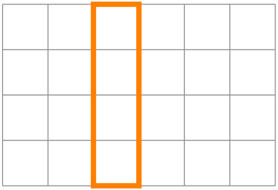

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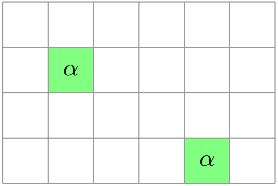

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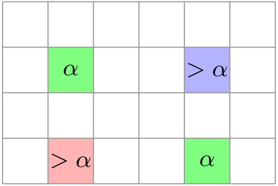

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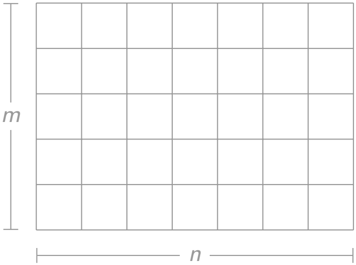
Bounds on $\chi_{\mathrm{os}}(K_m \,\square\, K_n)$

Lemma

In an ordered star coloring of $K_m \,\square\, K_n$, each column of height m contains k colors which are assigned to at most k vertices for $1 \leq k \leq m$.



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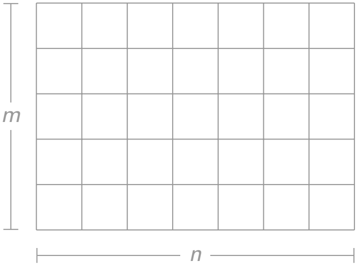

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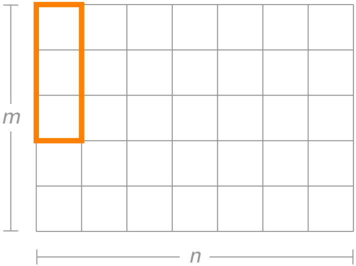

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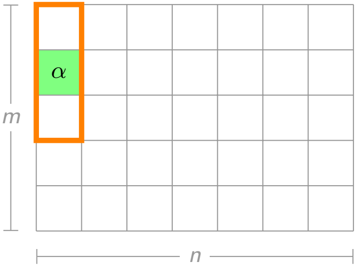

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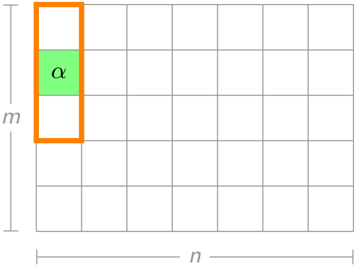

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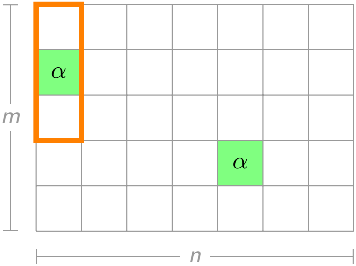

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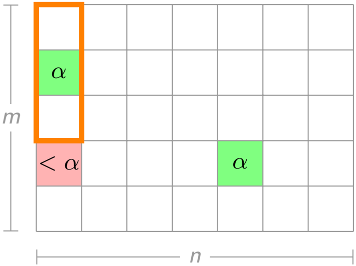

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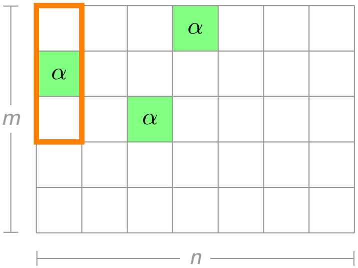

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- At most k vertices have color α.

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$\chi_{\mathrm{os}}(K_m \square K_n) \geq nH_m$, *where* $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$.

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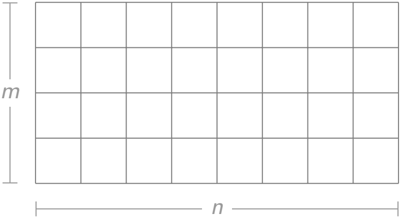
Corollary

If m *is fixed and* $n \to \infty$, *then* $\chi_{\mathrm{os}}(K_m \,\square\, K_n) = (1 + o(1))n \ln m$.

The diagonal case

Proposition

If $m \leq n$ and both are powers of 2, then $\chi_{\mathrm{os}}(K_m \,\Box\, K_n) \leq nm^{\lg 3 - 1}$.



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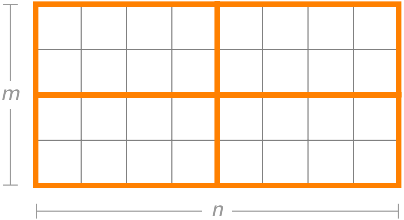

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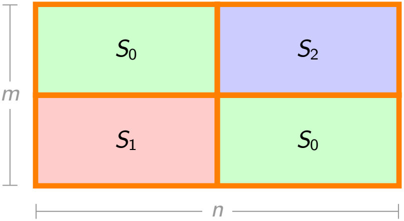

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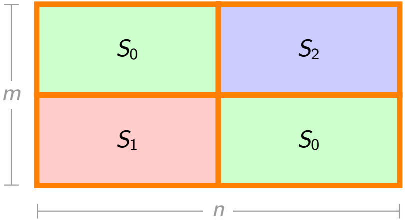

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- $\chi_{\mathrm{os}}(K_m \,\square\, K_n) \leq 3\chi_{\mathrm{os}}(K_{m/2} \,\square\, K_{n/2})$.

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- $\Omega(n \log n) \le \chi_{\text{os}}(K_n \,\square\, K_n) \le O(n^{1.585})$

Connection to advertised problem

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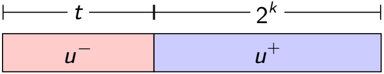

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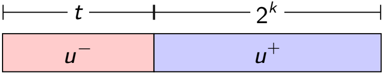

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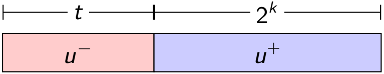

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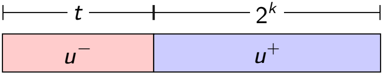

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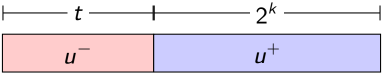

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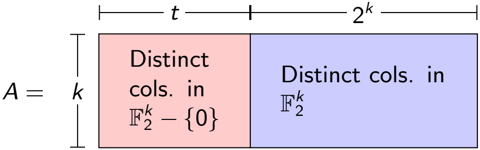

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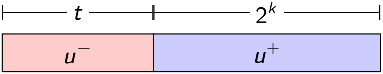

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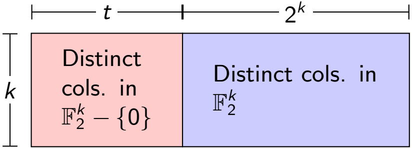

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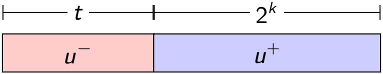

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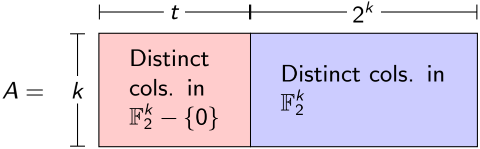

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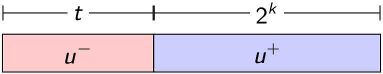

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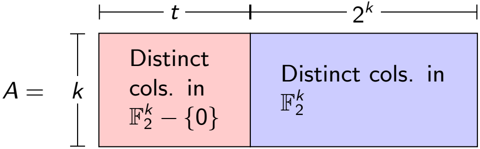

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