Cycle Spectra of Hamiltonian Graphs

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Abstract

We prove that every graph consisting of a spanning cycle plus p chords has cycles with more than $\sqrt{p} - \frac{1}{2}\ln p - 2$ different lengths. The result is asymptotically sharp in the sense that when $p = n^2/4 - n$ there are exactly $\sqrt{p+1}$ lengths of cycles in $K_{n/2,n/2}$. For general m and n, there are Hamiltonian graphs with n vertices and m edges having at most $2\left\lceil\sqrt{m-n+1}\right\rceil$ different cycle lengths.

Keywords: cycle, cycle spectrum, Hamiltonian graph, Hamiltonian cycle.

1 Introduction

The *cycle spectrum* of a graph G is the set of lengths of cycles in G. A cycle containing all vertices of a graph is a *spanning* or *Hamiltonian cycle*, and a graph having such a cycle is a *Hamiltonian graph*. An n-vertex graph is *pancyclic* if its cycle spectrum is $\{3, \dots, n\}$. Our graphs have no loops or multiple edges. A graph is k-regular if every vertex has degree k (that is, k incident edges).

Interest in cycle spectra arose from Bondy's "Metaconjecture" (based on [3]) that sufficient conditions for the existence of Hamiltonian cycles usually also imply that a graph is pancyclic, with possibly a small family of exceptions. For example, Bondy [3] showed that the sufficient condition on n-vertex graphs due to Ore [15] (the degrees of any two nonadjacent vertices sum to at least n) implies also that G is pancyclic or is the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. Schmeichel and Hakimi [12] showed that if a spanning cycle in an n-vertex

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graph G has consecutive vertices with degree-sum at least n, then G is pancyclic or bipartite or omits only $n-1$ from the cycle spectrum, the latter occurring only when the degree-sum is exactly n. Bauer and Schmeichel [1] used this to give unified proofs that the conditions for Hamiltonian cycles due to Bondy [4], Chvátal [5], and Fan [8] also imply that a graph is pancyclic, with small families of exceptions. Further results about the cycle spectrum under degree conditions on selected vertices in a spanning cycle appear in [9] and [13].

At the 1999 conference "Paul Erdős and His Mathematics", Jacobson and Lehel proposed the opposite question: *When sufficient conditions for spanning cycles are relaxed, how small can the cycle spectrum be if the graph is required to be Hamiltonian?* For example, consider regular graphs. Bondy's result [3] implies that $\lceil n/2 \rceil$-regular graphs other than $K_{\frac{n}{2},\frac{n}{2}}$ are pancyclic. On the other hand, 2-regular Hamiltonian graphs have only one cycle length. For $3 \le k \le \lceil n/2 \rceil - 1$, Jacobson and Lehel asked for the minimum size of the cycle spectrum of a k-regular n-vertex Hamiltonian graph, particularly when $k = 3$.

Let $s(G)$ denote the size of the cycle spectrum of a graph G. At the SIAM Meeting on Discrete Mathematics in 2002, Jacobson announced that he, Gould, and Pfender had proved $s(G) \ge c_k n^{1/2}$ for k-regular graphs with n vertices. Others later independently obtained similar bounds, without seeking to optimize c_k. For an upper bound, Jacobson and Lehel constructed the 3-regular example below with only $n/6+3$ distinct cycle lengths (when $n \equiv 0 \mod 6$ and $n > 6$), and they generalized it to the upper bound $\frac{n}{2}\frac{k-2}{k} + k$ for k-regular graphs.

Example 1 When $k = 3$ and 6 divides n (with $n > 6$), take $n/6$ disjoint copies of $K_{3,3}$ in a cyclic order, with vertex sets $V_1, \ldots, V_{n/6}$. Remove one edge from each copy and replace it by an edge to the next copy to restore 3-regularity. A cycle of length different from 4 or 6 must visit each V_i, and in each V_i it uses 4 or 6 vertices. Hence the cycle lengths are 4, 6, and each even integer from $2n/3$ through n. For the generalization, use $K_{k,k}$ instead of $K_{3,3}$. □

A related problem is the conjecture of Erdős [6] that $s(G) \ge \Omega\left(d^{\lfloor (g-1)/2 \rfloor}\right)$ when G has girth g and average degree d. Erdős, Faudree, Rousseau, and Schelp [7] proved the conjecture for $g = 5$. Sudakov and Verstraëte [14] proved the full conjecture in a stronger form, obtaining $\frac{1}{8}\left(d^{\lfloor (g-1)/2 \rfloor}\right)$ consecutive even integers in the cycle spectrum for graphs with fixed girth g and average degree $48(d+1)$. Gould, Haxell, and Scott [10] proved a similar result: for $c > 0$, there is a constant k_c such that for sufficiently large n, the cycle spectrum of every n-vertex graph G having minimum degree at least cn and longest even cycle length $2l$ contains all even integers from 4 up to $2l - k_c$ (see also [2]).

Prior arguments for lower bounds on $s(G)$ when G is regular and Hamiltonian used only the number of edges, not regularity. Suppose that G has n vertices and m edges. The coefficient c in a general lower bound of the form $s(G) \ge \sqrt{c(m-n)}$ cannot exceed 1, since $s(K_{\frac{n}{2},\frac{n}{2}}) = \sqrt{m-n+1}$. We give a construction for $m \le n^2/4$ that is far from regular.

Example 2 For $t \le n/2$, form a graph G by replacing one edge of $K_{t,t}$ with a path having $n-2t$ internal vertices; there are n vertices and m edges, where $m = t^2 - 2t + n \le n^2/4$. The cycle spectrum of G consists of the $t-1$ even numbers in $\{4, \dots, 2t\}$ and the $t-1$ numbers from $n-2t+4$ to n having the same parity as n. Thus $s(G) \le 2(t-1) = 2\sqrt{m-n+1}$. Equality holds when $t \le \lceil n/4 \rceil$, but when $\lceil n/4 \rceil < t \le n/2$ and n is even the two sets of $t-1$ numbers overlap (the two sets are the same when $m = n^2/4$, reducing to $s(K_{\frac{n}{2},\frac{n}{2}}) = \sqrt{m-n+1}$).

Deleting edges cannot enlarge the cycle spectrum. Hence in general we can let $t = \lceil \sqrt{m-n+1} \rceil + 1$, apply the construction above for n and t, and discard edges to wind up with m edges and $s(G) \le 2\lceil \sqrt{m-n+1} \rceil$.

For a graph G with $m > n^2/4$, we have $s(G) \le n-2 \le 2\sqrt{m-n+1}$. In fact, the main result of [3] implies that every Hamiltonian graph with more than $n^2/4$ edges is pancyclic. Thus $s(G) = n-2$ when $m > n^2/4$.

Possibly this construction usually has the smallest spectrum among Hamiltonian graphs with n vertices and m edges. However, when $(n, m) = (14, 21)$, the Heawood graph has smaller spectrum than the graph constructed in this way. □

Our main result is $s(G) > \sqrt{p} - \frac{1}{2}\ln p - 2$ when G consists of an n-cycle with p chords.

2 The Lower Bound

A path with endpoints x and y is an *x, y-path*. A *chord* of a path (or cycle) P in a graph is an edge of the graph not in P whose endpoints are in P, and the *length* of the chord is the distance in P between its endpoints. Given a path P with vertices $v_1, \dots, v_n$ in order, two chords $v_a v_c$ and $v_b v_d$ *overlap* if $a < b < c < d$.

Lemma 3 *If a graph G consists of an x, y-path P and h pairwise-overlapping chords of length l, then G contains x, y-paths having at least $h-1$ distinct lengths. Having only $h-1$ lengths requires l odd, $h \ge (l+3)/2$, and chords starting at h consecutive vertices along P.*

Proof. For $h \le 2$, there are at least h distinct lengths, so we may assume $h \ge 3$. Let n be the length of P. Let $e_1, \dots, e_h$ be the chords in the order in which they are encountered along P from x to y. Let d_i be the distance along P from the first endpoint of e_{i-1} to the first endpoint of e_i, for $2 \le i \le h$.

Let $P_{i,j}$ be the unique x, y-path using exactly two chords e_i and e_j, along with edges of P. Let p_j be the length of $P_{1,j}$, for $2 \le j \le h$. Note that $p_j = p_{j-1} - 2d_j$ for $3 \le j \le h$. The $h-1$ paths $P_{1,2}, \dots, P_{1,h}$ have distinct lengths, which proves the first statement.

The length of $P_{1,2}$ is $n - 2d_2 + 2$. Thus the full path P provides an additional length unless $d_2 = 1$. If $d_j > 1$ for any larger j, then the length of $P_{2,j}$ is strictly between p_{j-1} and p_j. Hence an extra length arises unless the chords start at consecutive vertices along P.

The $h-1$ lengths we have found are $n, n-2, \ldots, n-2h+4$. The length of any x,y-path that uses exactly one chord is $n - l + 1$. To avoid generating a new length, it must be that l is odd and $2h - 4 \geq l - 1$. □

Definition 4 Let G be a graph consisting of an n-cycle C plus q chords of length l, where $l < n/2$. Say that a chord *covers* the edges and the internal vertices of the path of length l along C joining its endpoints. Specify a forward direction along C. Let $C[x,y]$ denote the subpath of C traversed by moving from x to y along C in the forward direction. When uv is a chord of length l and $C[u,v]$ has length l, we say that u is the *start* and v is the *end* of uv. For a chord e, let $F(e)$ be the set consisting of e and all chords that cover the end of e.

Select a chord e_1 so that $|F(e_1)| \geq |F(e)|$ for every chord e. For $j > 1$, let e_j be the first chord encountered in the forward direction from e_{j-1} that does not overlap e_{j-1} or e_1; if no such chord exists, then stop and set $\alpha = j-1$. Note that $F(e_i) \cap \{e_1, \ldots, e_\alpha\} = \{e_i\}$ for each i and that the sets $F(e_1), \ldots, F(e_\alpha)$ are pairwise disjoint. The selected edges $\{e_1, \ldots, e_\alpha\}$ form a *greedy chord system* for G (see Figure 1, which also includes notation used in Theorem 5).

Given a greedy chord system starting with e_1, let v_1 be the start of e_1. Let the vertices of C be $v_1, \ldots, v_n$ in order in the forward direction. In the case $\alpha \geq 2$, let z be the end of e_2, and say that a cycle in G is *long* if it contains $C[z, v_1]$ and has length at least $n - 2(l-1) + 1$.

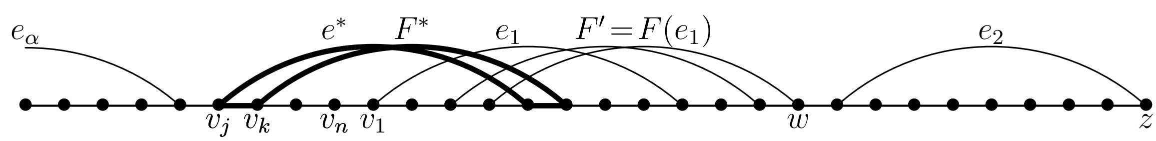

Figure 1: A greedy chord system

From a greedy chord system, we will build a large family of cycles with distinct lengths by combining cycles of length at most $n - \alpha(l-1) + 2$, intermediate-length cycles, and long cycles. The intermediate-length cycles are formed from the long cycles by replacing portions of C with chords.

Theorem 5 *Let G be a graph consisting of an n-cycle C plus q chords of length l, where $l < n/2$. The size $s(G)$ of the cycle spectrum of G is at least $(q-3)/2$ when l is even and at least $(q - 3 - \frac{q}{l})/2$ when l is odd.*

Proof. Consider a greedy chord system $e_1, \ldots, e_\alpha$. Let $F' = F(e_1)$. Let w be the end of the chord in F' that overlaps e_1 the least. Let F^* be the set of chords not in $\bigcup_{i=1}^{\alpha} F(e_i)$; since none of these chords overlaps e_α, each overlaps e_1. If $F^* \neq \varnothing$, then let e^* be the first chord of F^* after e_α in the forward direction (see Figure 1).

When $\alpha = 1$, we have $|F'| + |F^*| = q$. If also $F^* = \varnothing$, then $|F'| = q$; otherwise, $F^* \subseteq F(e^*)$. Hence $|F'| \geq \lceil q/2 \rceil$. Lemma 3 yields v_1, w-paths of at least $|F'| - 1$ lengths that combine with $C[w, v_1]$ to form cycles of $|F'| - 1$ distinct lengths. Hence we may assume $\alpha \geq 2$.

Using F^*, we now obtain $(|F^*| - 1)/2$ short cycle lengths. We may assume $|F^*| \geq 2$. Let j be the index of the start of e^*; that is, $e^* = v_j v_{j+l-n}$. Through each chord $v_k v_{k+l-n}$ in $F^* - \{e^*\}$, we consider two cycles. One uses $v_k v_{k+l-n}$ and e^* and the two paths $C[v_j, v_k]$ and $C[v_{j+l-n}, v_{k+l-n}]$ with length $k - j$ (see Figure 1). The other uses $v_k v_{k+l-n}$ and e_1 and the two paths $C[v_k, v_1]$ and $C[v_{k+l-n}, v_{1+l}]$ with length $n - k + 1$. The lengths of the cycles are $2(k - j + 1)$ and $2(n - k + 2)$; their minimum is at most $n - j + 3$.

Thus we obtain $|F^*| - 1$ cycles having length at most $n - j + 3$, with each such length occurring at most twice. We conclude that the spectrum contains at least $(|F^*| - 1)/2$ values bounded by $n - j + 3$. The index of the end of e_α is at least $1 + \alpha(l - 1)$. Hence $j \geq 1 + \alpha(l - 1)$, and the lengths of the short cycles are bounded by $n - \alpha(l - 1) + 2$.

From the long cycles in G we now construct cycles of intermediate lengths; let ρ be the number of distinct lengths of long cycles. Since long cycles contain $C[z, v_1]$, they contain all edges of C covered by any of $e_3, \ldots, e_\alpha$. These chords can be used to replace portions of long cycles. Each such replacement yields cycles of ρ distinct lengths, shorter by $l - 1$ than the lengths we previously had. Since the long cycles have length at least $n - 2(l - 1) + 1$, performing this shift $\alpha - 2$ times produces $\alpha - 1$ sets of size ρ. Values are generated at most twice; the original values exceeding $n - (l - 1)$ and the values that are at most $n - (\alpha - 1)(l - 1)$ appear only once. Repeating these ρ values yields a list of $\alpha\rho$ values in which each value appears at most twice. Hence we obtain at least $\alpha\rho/2$ cycle lengths that are all at least $n - \alpha(l - 1) + 1$. Since the short cycle lengths are all even, at most one of the short lengths is repeated in this set.

The greedy choice of e_1 yields $|F'| \geq \left\lceil \frac{q - |F^*|}{\alpha} \right\rceil$. To obtain a useful lower bound on $\alpha\rho/2$, we compare ρ to $|F'|$. Let G' be the induced subgraph of G consisting of $C[v_1, w]$ and the chords in F'. Since the chords in F' are pairwise overlapping, Lemma 3 yields v_1, w-paths in G' with $|F'| - 1$ distinct lengths. Furthermore, there are at least $|F'|$ distinct lengths unless l is odd, $|F'| \geq (l + 3)/2$, and the starts of the chords in F' are consecutive along C.

If $w = v_{l+1}$, then the greedy choice of e_1 implies that the chords are pairwise noncrossing and $s(G) = q + 1$. We may thus assume $w \neq v_{l+1}$, so every v_1, w-path in G' has length at least 2. Adding $C[w, v_1]$ to v_1, w-paths of distinct lengths in G' creates cycles of distinct lengths in G. Since at least $n-2l+1$ edges of C are not in G', these cycles are long cycles.

Thus when l is even, we have shown that $\rho \geq |F'|$. Hence

$$s(G) \geq \frac{\alpha\rho}{2} + \frac{|F^*|-1}{2} - 1 \geq \frac{q-|F^*|}{2} + \frac{|F^*|-1}{2} - 1 = \frac{q-3}{2}.$$

If l is odd, then $\rho \geq \left\lceil \frac{q-|F^*|}{\alpha} \right\rceil$ still holds if $|F'| > \left\lceil \frac{q-|F^*|}{\alpha} \right\rceil$. Since always $|F'| \geq \left\lceil \frac{q-|F^*|}{\alpha} \right\rceil$, we may assume that equality holds. If $\rho \geq |F'|$ does not hold, then Lemma 3 implies that $|F'| \geq (l+3)/2$ and that the chords in F' are consecutive. Hence the lengths of the long cycles are $n, n-2, \ldots, n-2\,|F'|+4$. We consider two cases, depending on whether e_2 overlaps some chord in F'.

Case 1: *e_2 overlaps no chord in F'.* Here e_2, like $e_3, \ldots, e_\alpha$, can be used to reduce cycle lengths by $l-1$. Since $|F'| \geq (l+3)/2$, the long cycle lengths include $n, n-2, \ldots, n-(l-1)$; there are $(l+1)/2$ of them. Hence shifting the values down by $l-1$ leaves no gaps. After using each of $e_2, \ldots, e_\alpha$ to reduce the lengths by $l-1$, we obtain $1+\alpha(l-1)/2$ consecutive lengths of the same parity. Omitting the smallest, we have $\frac{1}{2}\alpha(l-1)$ cycle lengths, each at least $n-\alpha(l-1)+2$.

If $\alpha \geq q/l$, then $\frac{1}{2}\alpha(l-1) \geq \frac{1}{2}q(1-\frac{1}{l}) \geq \frac{1}{2}(q-|F^*|-\frac{q}{l})$. If $\alpha < q/l$, then we use $l \geq |F'| = \left\lceil \frac{q-|F^*|}{\alpha} \right\rceil$ to compute

$$\frac{1}{2}\alpha(l-1) \geq \frac{1}{2}(|F'|-1)\alpha \geq \frac{1}{2}(q-|F^*|-\alpha) > \frac{1}{2}\left(q-|F^*|-\frac{q}{l}\right).$$

Length $n-\alpha(l-1)+2$ may also be counted among the short cycle lengths. Adding the $(|F^*|-1)/2$ short lengths and subtracting 1 for the possible overlap yields at least the desired number of lengths.

Case 2: *e_2 overlaps some chord in F'.* Since the chords in F' are consecutive, this case requires that the start of e_2 is just before the end of some chord e' in F'. Let v' be the start of e'. The cycle that uses these two chords, the edge they both cover, and $C[z, v']$ has length $n-2(l-1)+2$; hence it is a long cycle. We obtain $\rho \geq |F'|$ unless this length already appears among those we have generated, which requires $2\,|F'|-4 \geq 2(l-1)-2$, so $|F'| \geq l$. Since $|F'| \leq l$, equality holds.

As noted above, we have $n, n-2, \ldots, n-2(l-2)$ as $l-1$ distinct cycle lengths, all long. Lowering the bottom half of them by $l-1$ exactly $\alpha-2$ times yields $\frac{1}{2}\alpha(l-1)$ distinct cycle lengths. The least of them is $n-\alpha(l-1)+2$. This is exactly the same situation we obtained in Case 1, so the same computation completes the proof. □

Theorem 6 *If G is an n-vertex Hamiltonian graph with m edges, then $s(G) > \sqrt{p} - \frac{1}{2}\ln p - 2$, where $p = m-n$.*

Proof. Let C be a spanning cycle in G. Let L be the set of lengths of chords of C in G, and let $t = |L|$. For each $l \in L$, we obtain two lengths of cycles in G; they are $l+1$ and $n-l+1$ if $l < n/2$ (using one chord of length l), and they are $n/2+1$ and n if $l = n/2$. Hence $s(G) \geq 2t$, which suffices if $t \geq \frac{1}{2}\sqrt{p}$. We may therefore assume that $2t < \sqrt{p}$.

For $l \in L$, let q_l be the number of chords of length l. By Theorem 5, when $l < n/2$ there are at least $\frac{l-1}{2l}q_l - \frac{3}{2}$ lengths of cycles using only edges of C and chords of length l. The lower bound also holds when $l = n/2$, since then the chords are pairwise overlapping and Lemma 3 applies, and always $q - 1 > \frac{l-1}{2l}q_l - \frac{3}{2}$.

We may assume that $\frac{l-1}{2l}q_l - \frac{3}{2} \leq \sqrt{p} - \frac{1}{2}\ln p - 2$ for odd $l \in L$, and $\frac{1}{2}q_l - \frac{3}{2} \leq \sqrt{p} - \frac{1}{2}\ln p - 2$ for even $l \in L$. Thus $q_l \leq (\sqrt{p} - \frac{1}{2}\ln p - \frac{1}{2})c_l$, where $c_l = 2$ when l is even and $c_l = 2 + \frac{2}{l-1}$ when l is odd. We obtain a contradiction by showing that these bounds on q_l sum to less than p. In light of the form of c_l, it suffices to prove this when all values in L are odd. The bound is now the worst when L consists of the first t positive odd numbers. We compute

$$\begin{aligned} p = \sum_{l \in L} q_l \leq \sum_{l \in L} \left(\sqrt{p} - \tfrac{1}{2}\ln p - \tfrac{1}{2}\right)\left(2 + \frac{2}{l-1}\right) \ &\leq \ \left(\sqrt{p} - \tfrac{1}{2}\ln p - \tfrac{1}{2}\right)\left[2t + \sum_{i=1}^{t}\frac{1}{i}\right] \\ < \left(\sqrt{p} - \tfrac{1}{2}\ln p - \tfrac{1}{2}\right)\left[\sqrt{p} + (1 + \ln t)\right] \ &< \ \left(\sqrt{p} - \tfrac{1}{2}\ln p - \tfrac{1}{2}\right)\left[\sqrt{p} + \tfrac{1}{2}\ln p + (1 - \ln 2)\right] \\ = p - \tfrac{1}{4}(\ln p)^2 - (\ln 2 - \tfrac{1}{2})\sqrt{p} - \tfrac{1}{4}(3 - \ln 4)\ln p - \tfrac{1}{2}(1 - \ln 2) \ &< \ p. \end{aligned}$$

The contradiction completes the proof. $\square$

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