Cycle Spectra of Hamiltonian Graphs

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Conjecture

$\exists c$: if G is a Hamiltonian subgraph of $K_{n,n}$ with $\delta(G) \geq c\sqrt{n}$, then $S(G) = \{4, 6, \ldots, 2n\}$.

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- Lower bounds on $s(G)$ when G is Hamiltonian and k-regular.
- In particular, what about $k = 3$?

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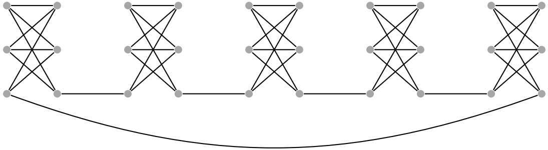

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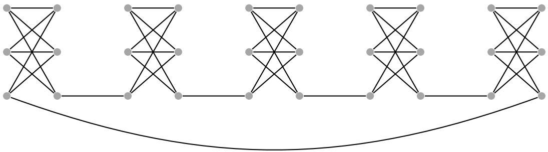

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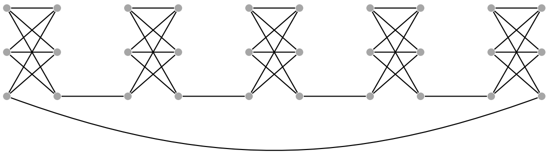

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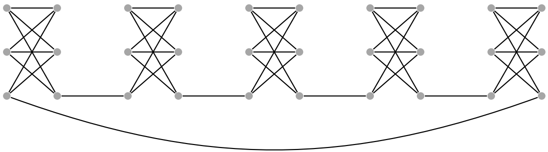

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- $s(G) = n/6 + 3$
- Generalizes to provide k-regular Hamiltonian graphs with $s(G) = \frac{k-2}{2k}n + k$ when $2k$ divides n.

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Theorem (Entringer–Schmeichel (1988))

If G is an n-vertex bipartite Hamiltonian graph with m edges and $m > n^2/8$, then G is bipancyclic (has cycles of all even lengths from 4 to n).

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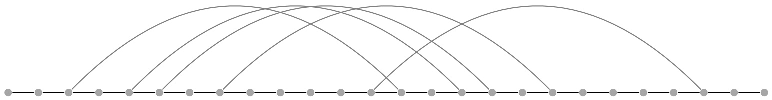

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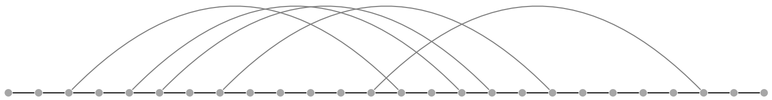

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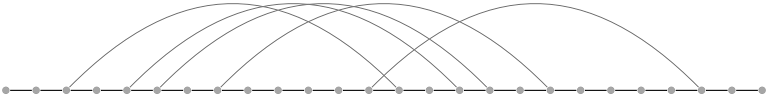

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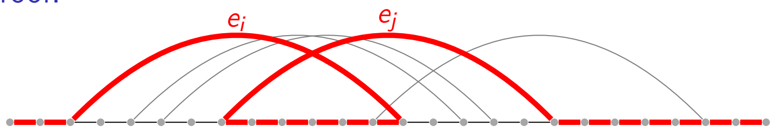

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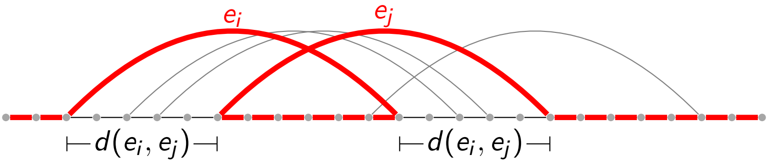

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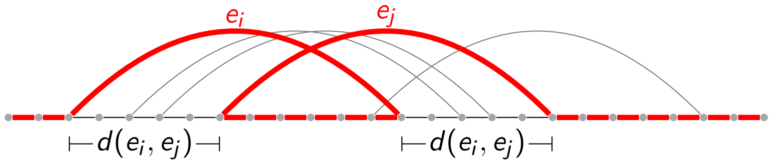

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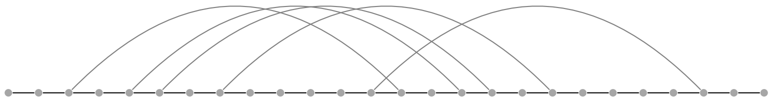

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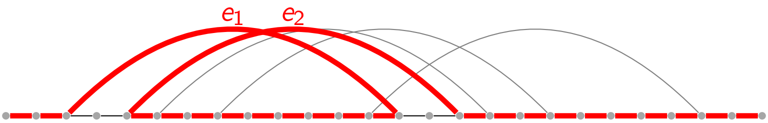

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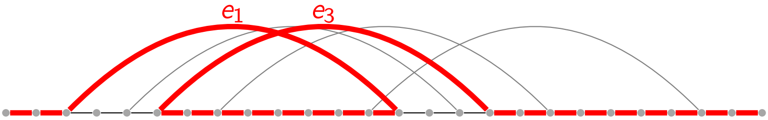

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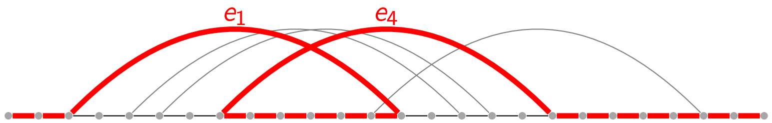

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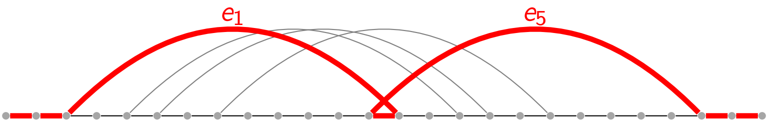

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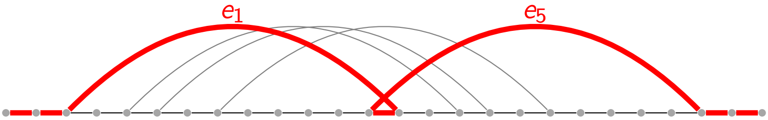

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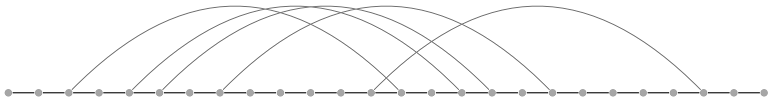

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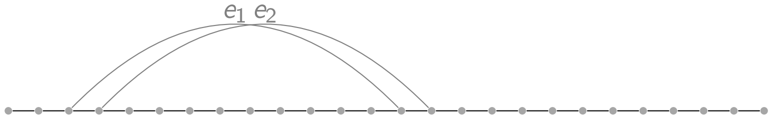

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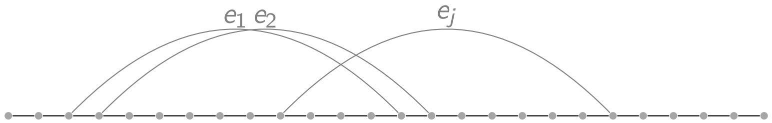

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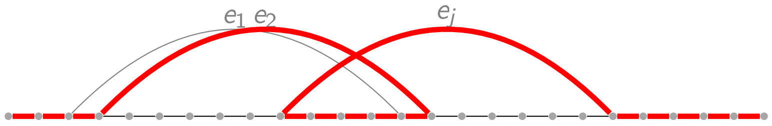

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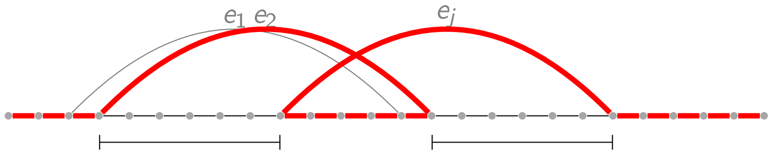

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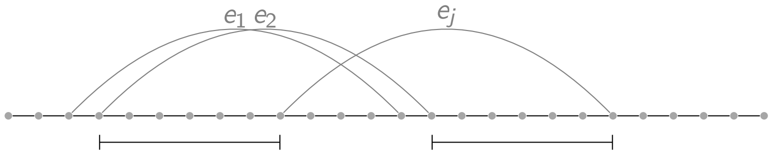

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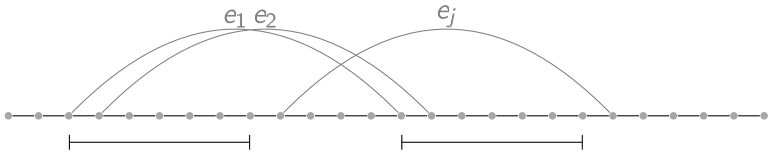

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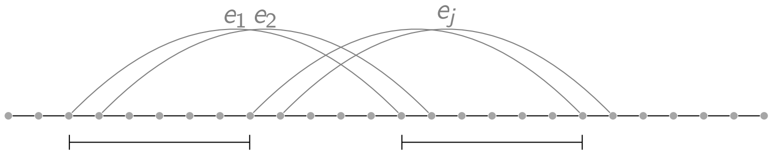

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- The length of $P_{2,j}$ ...
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- So, there is a chord immediately preceding e_j.

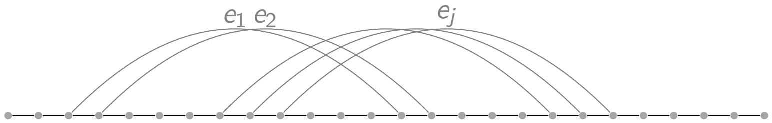
Overlapping chords lemma

Lemma

G: an x, y-path P plus h pairwise-overlapping chords of length ℓ. Then G contains x, y-paths of $h - 1$ distinct lengths. Having only $h - 1$ lengths requires that

1. *the chords are consecutive along P, and*
2. *ℓ is odd and $h \geq (\ell + 3)/2$.*

Proof.



- Consider a chord e_j.
- The length of $P_{2,j}$ ...
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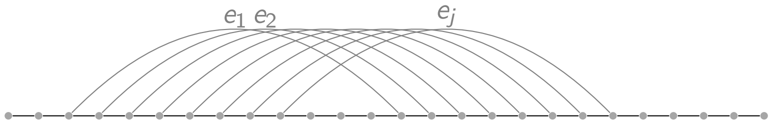

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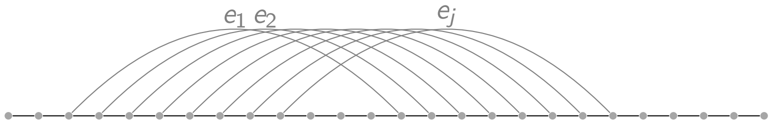

- Lengths of paths: $n, n-2, \ldots, n-2(h-2)$.

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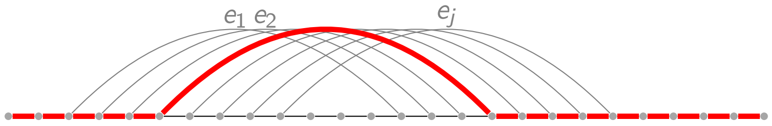

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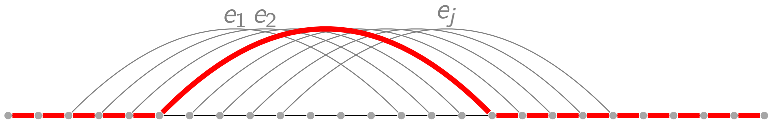

- Lengths of paths: $n, n-2, \ldots, n-2(h-2)$.
- Path with a single chord: length $n-(\ell-1)$.
- So $\ell - 1 \in \{0, 2, \ldots, 2(h-2)\}$.

Greedy chord system

- G: Hamiltonian cycle C plus q chords of length ℓ

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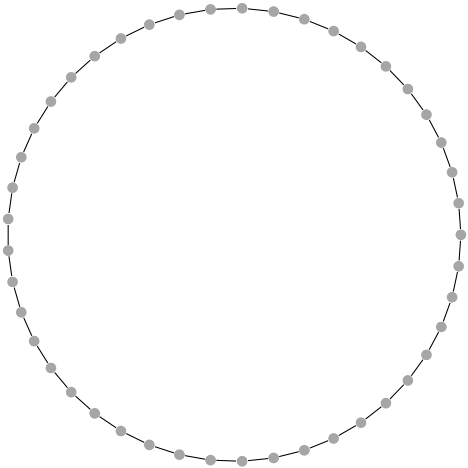

- Choose a forward direction along C.

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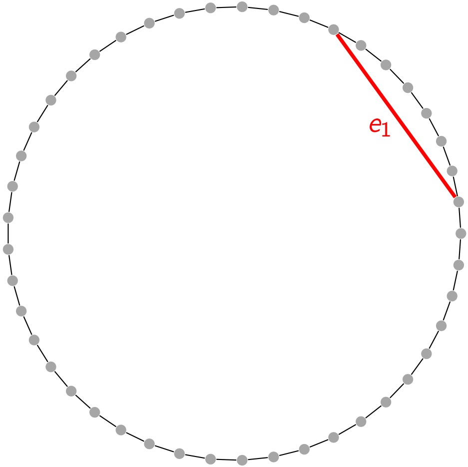

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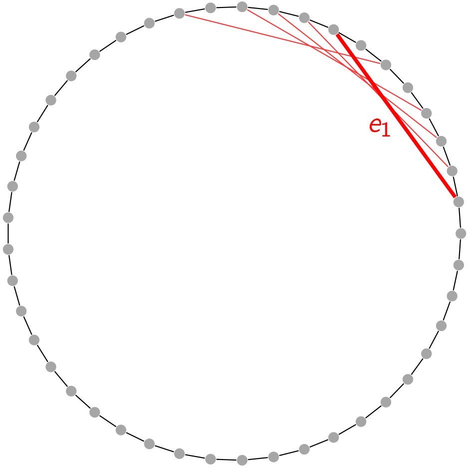

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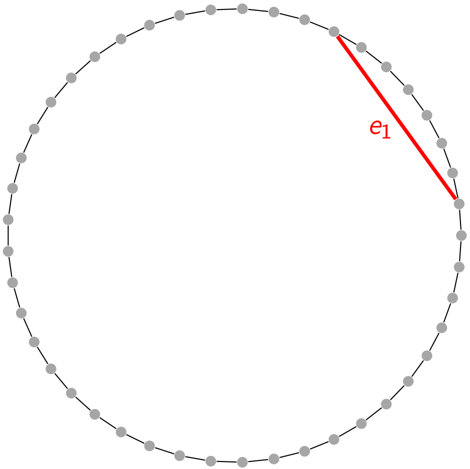

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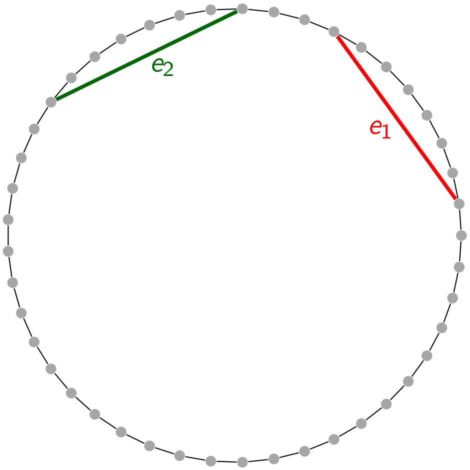

- Choose a forward direction along C.
- e_1: chord with most overlapping chords going forward.
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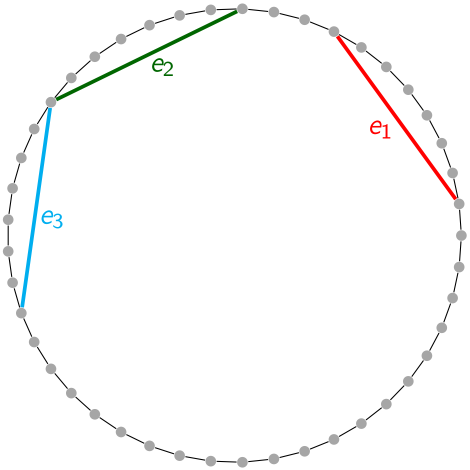

- Choose a forward direction along C.
- e_1: chord with most overlapping chords going forward.
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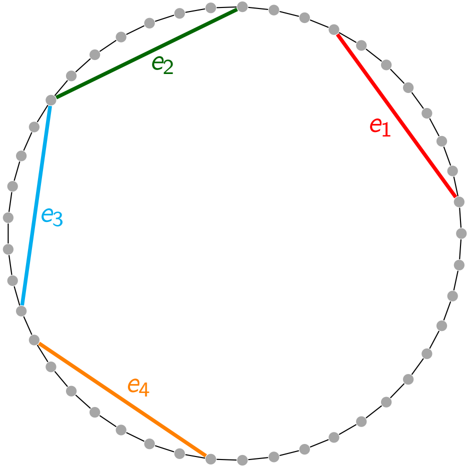

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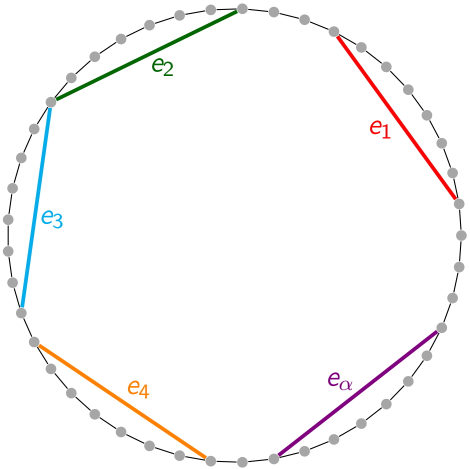

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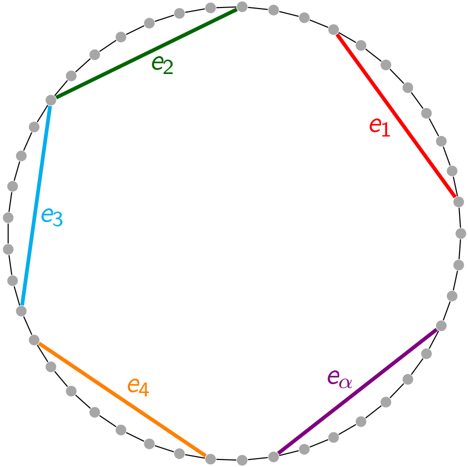

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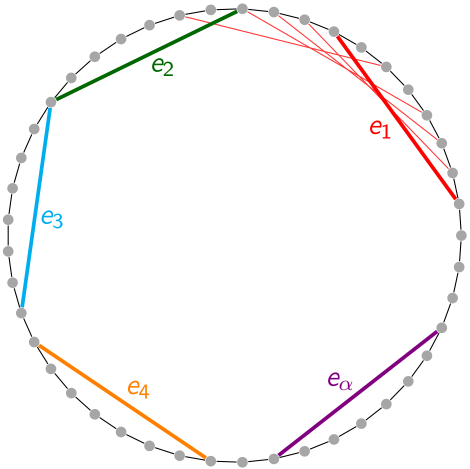

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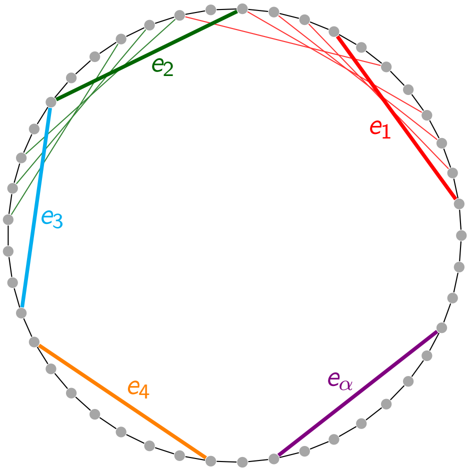

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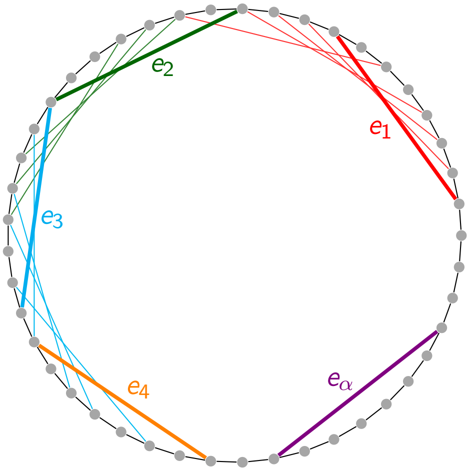

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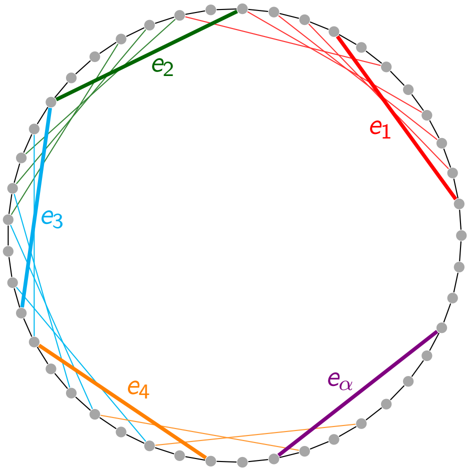

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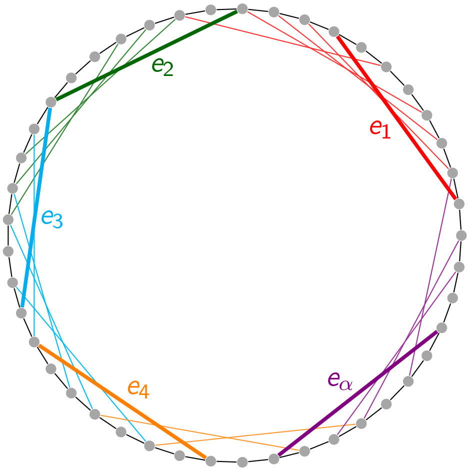

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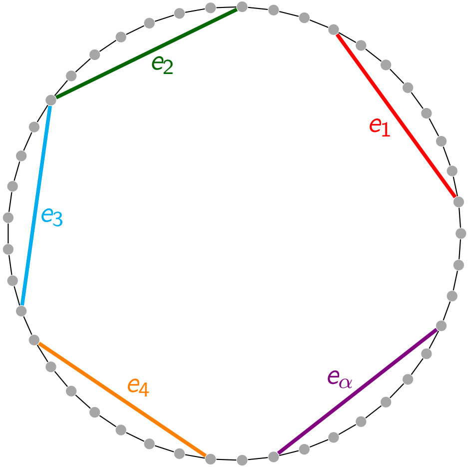

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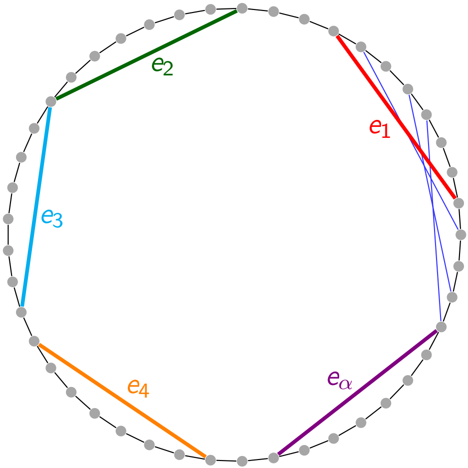

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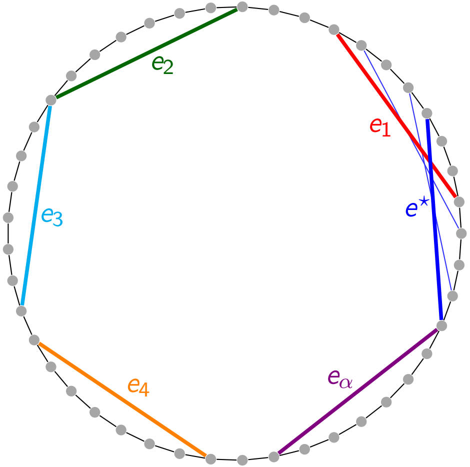

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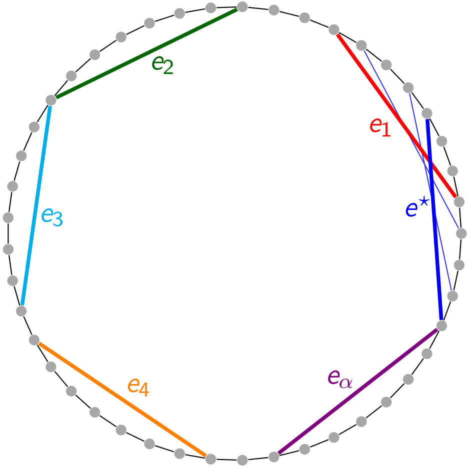

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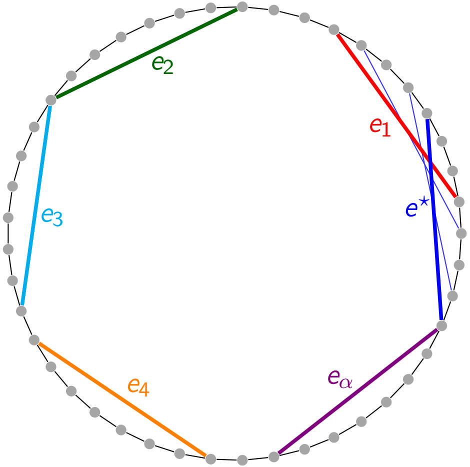

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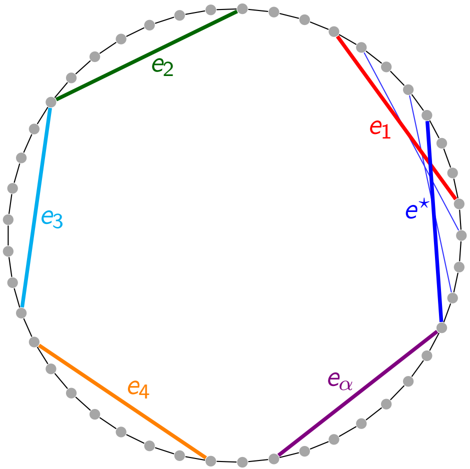

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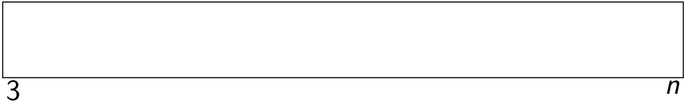
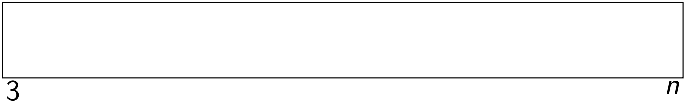
- Also: $|F_1| \geq |F^\star|$.

Spectrum bands



- We find many cycle lengths by dividing the space of possible cycle lengths into bands.

Spectrum bands



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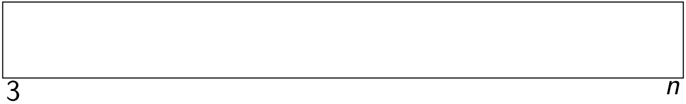

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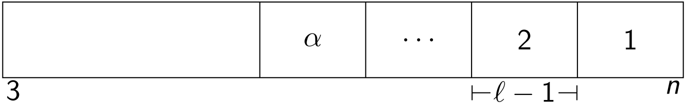

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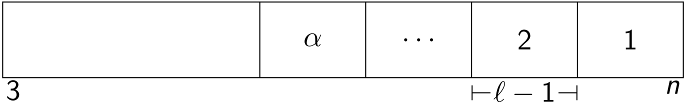
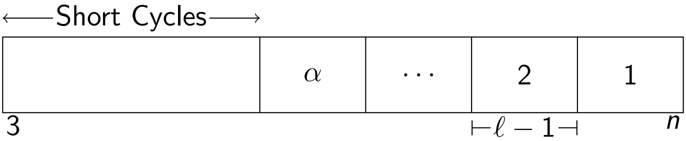
- We have α bands at the top, each of size $\ell - 1$.

Spectrum bands



- The jth band: from $n - j(\ell - 1) + 1$ to $n - (j - 1)(\ell - 1)$.

Spectrum bands



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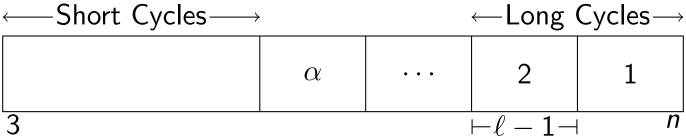

- The jth band: from $n - j(\ell - 1) + 1$ to $n - (j - 1)(\ell - 1)$.
- The short cycles: lengths below the top α bands.
- The long cycles: lengths in the top 2 bands.

Short cycles lemma

Lemma

If $\alpha \geq 2$, then G has short cycles of at least $\frac{|F^\star|-1}{2}$ distinct lengths.

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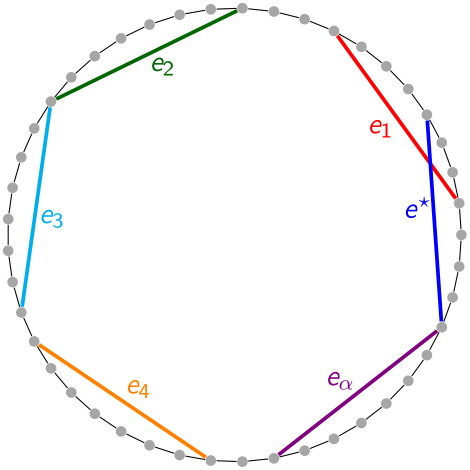

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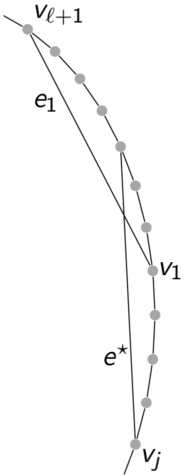

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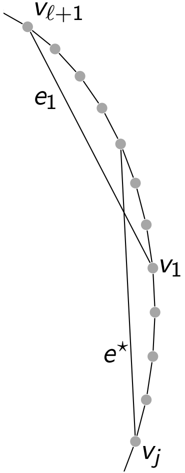

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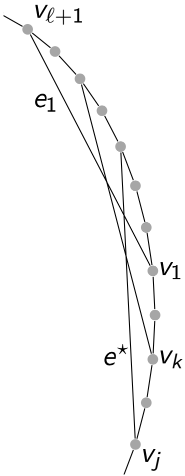

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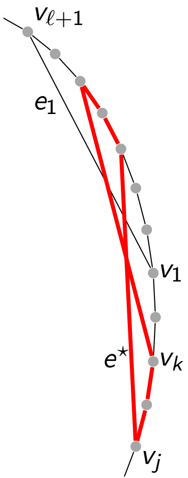

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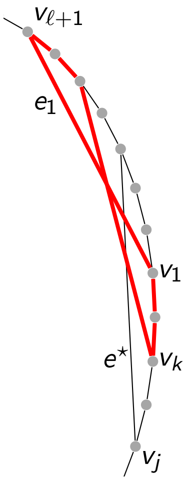

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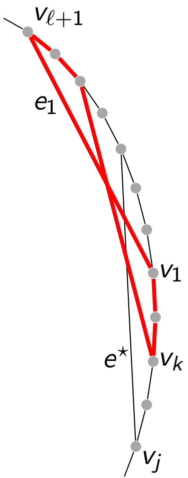

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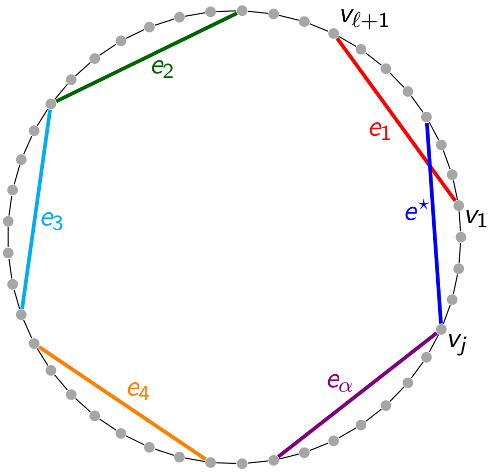

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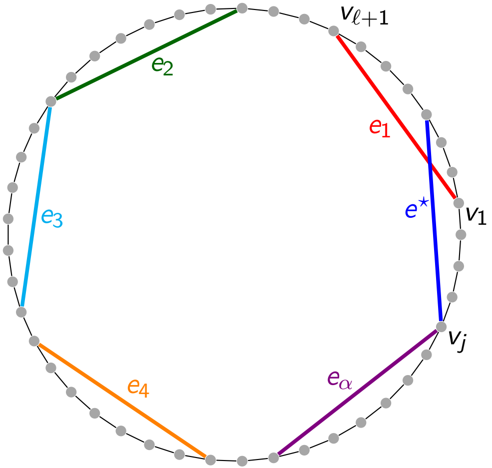

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- Consider a chord $e \in F^\star$ with $e \neq e^\star$.
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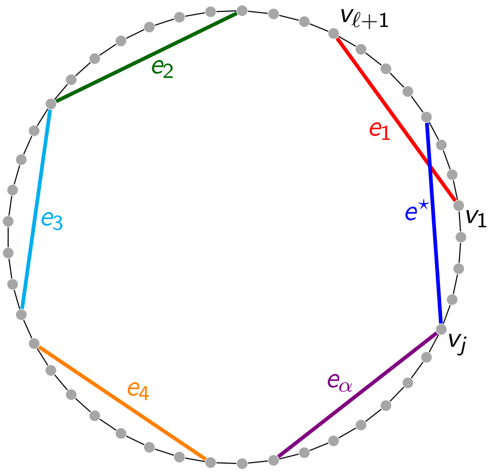
If $\alpha \geq 2$, then G has short cycles of at least $\frac{|F^\star|-1}{2}$ distinct lengths.



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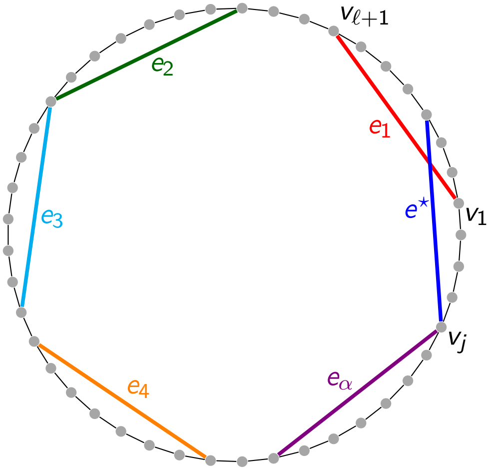

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- $\alpha \geq 2$: this cycle is short.

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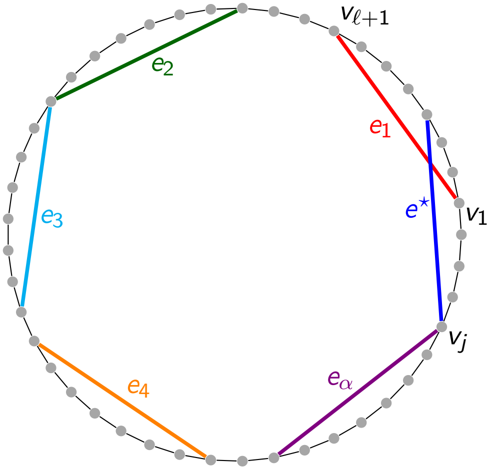

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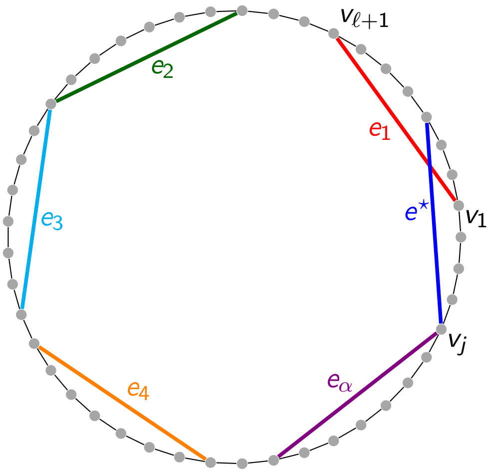

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Longer cycles

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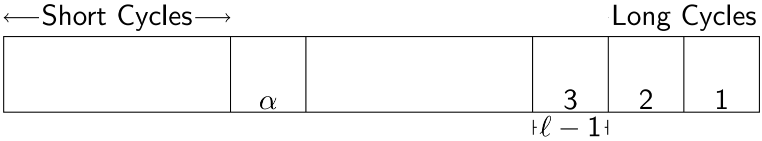

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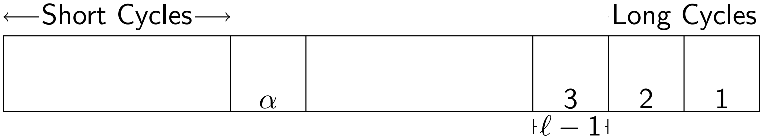

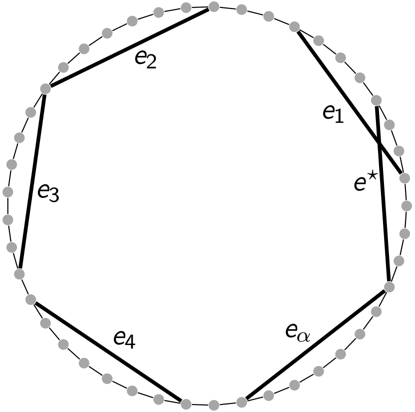

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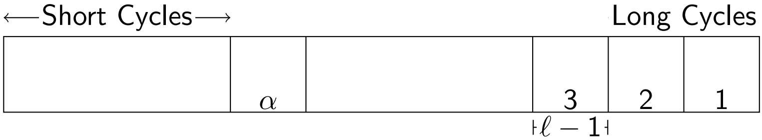

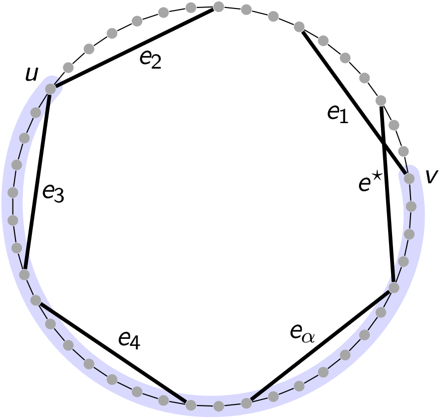

- A long cycle is good if it contains $C[u, v]$.

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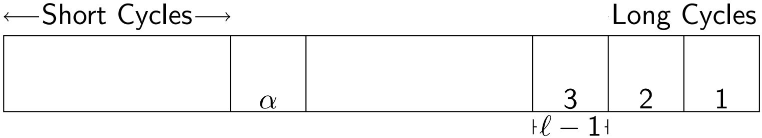

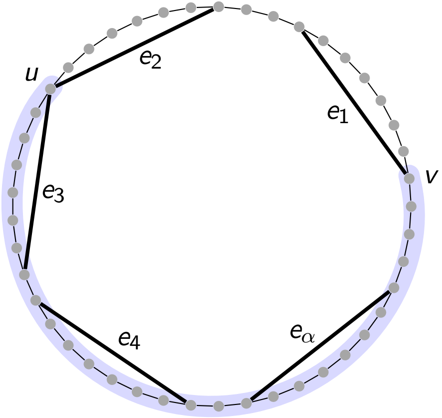

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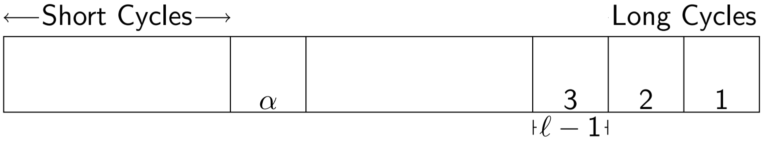

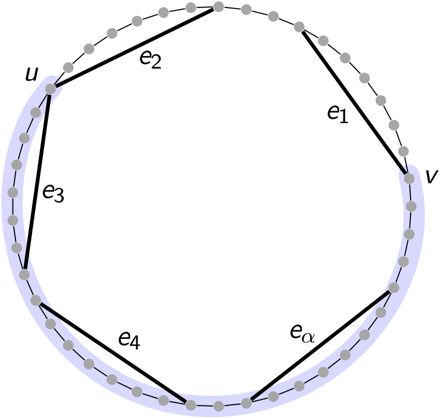

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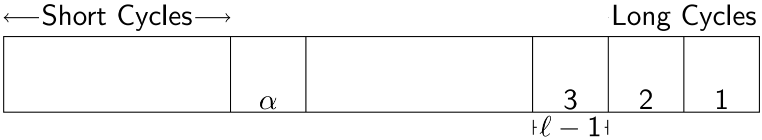

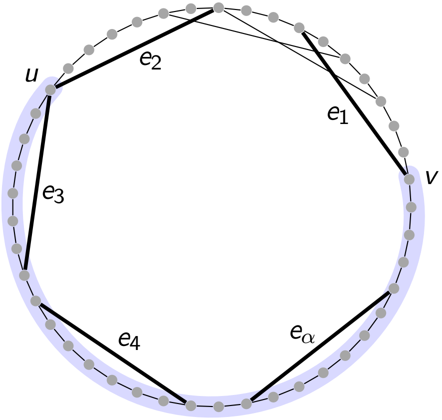

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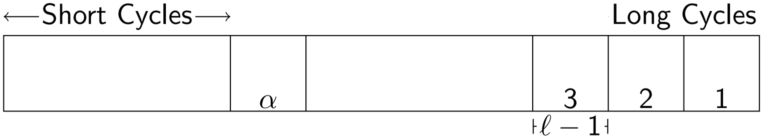

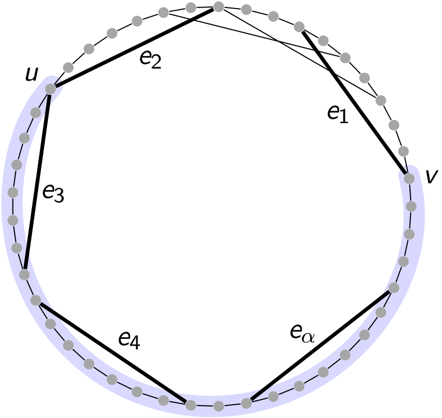

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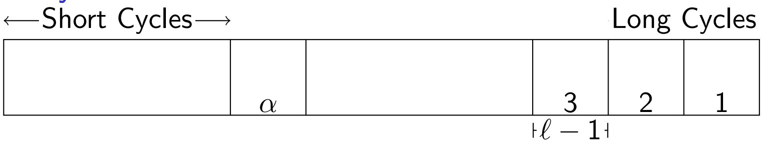

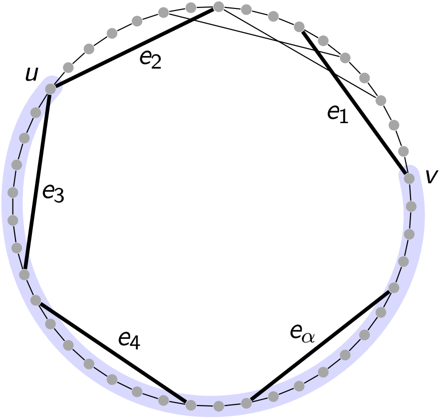

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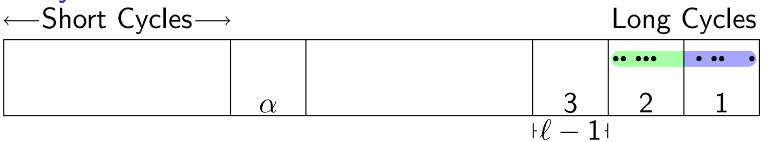

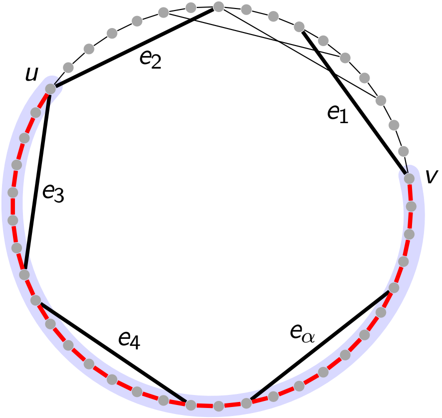

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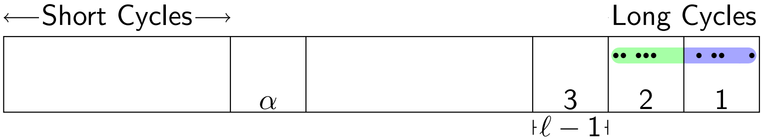

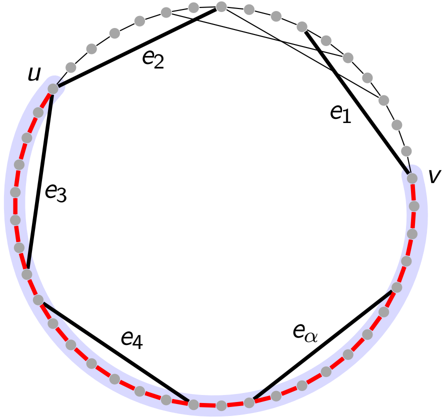

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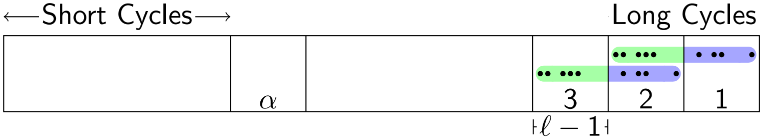

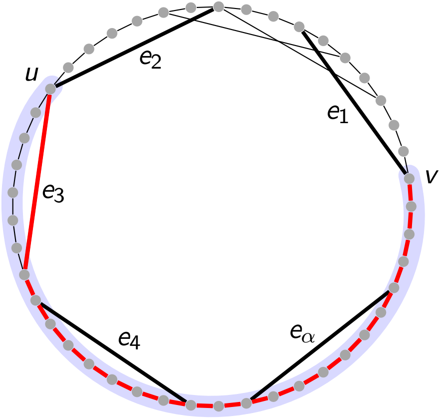

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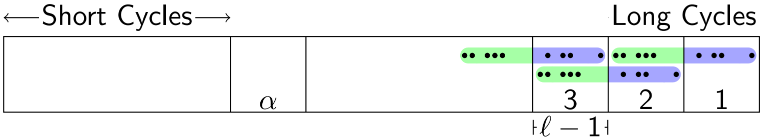

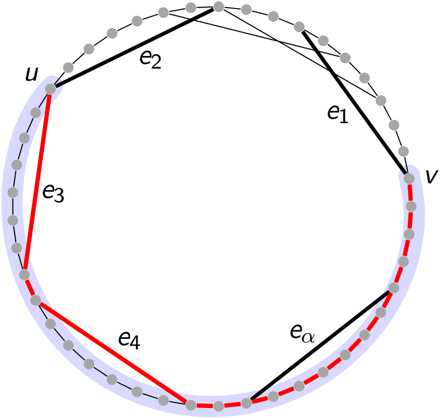

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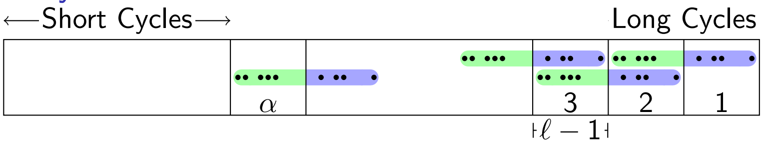

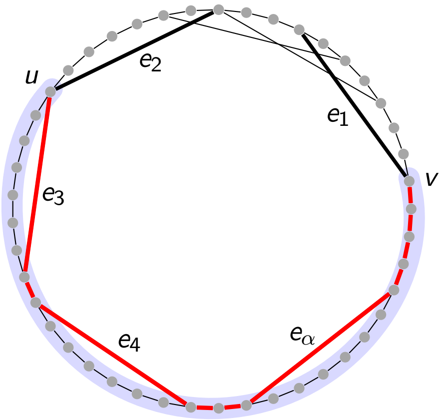

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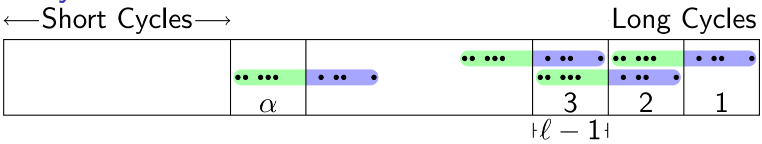

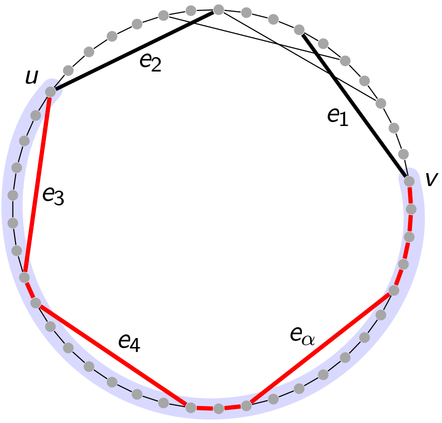

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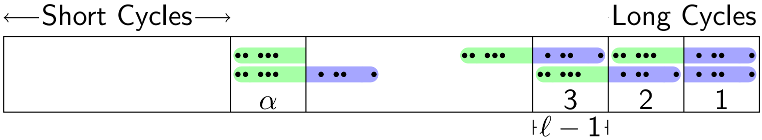

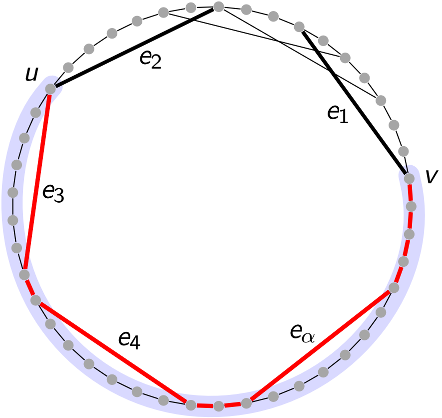

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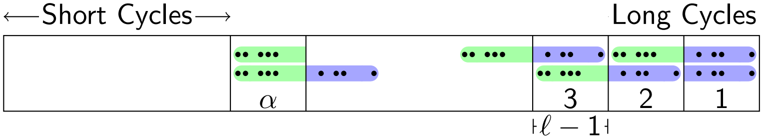

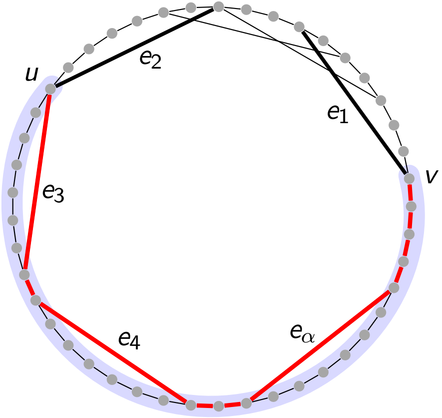

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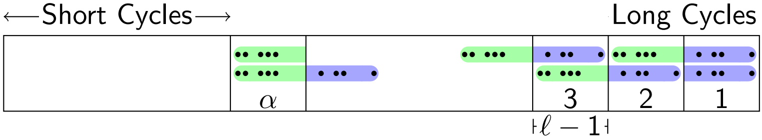

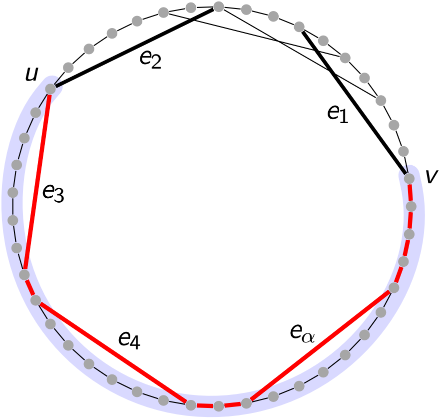

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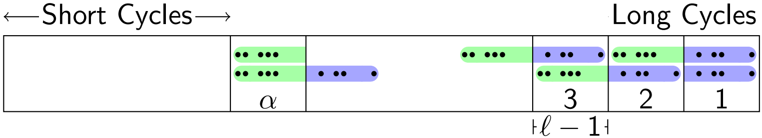

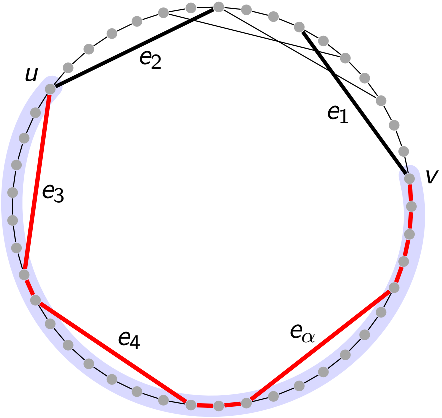

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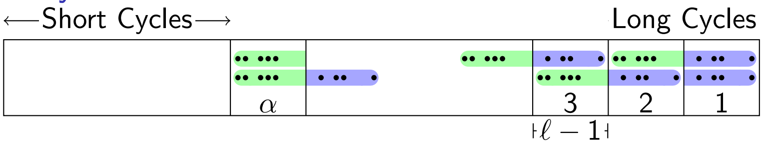
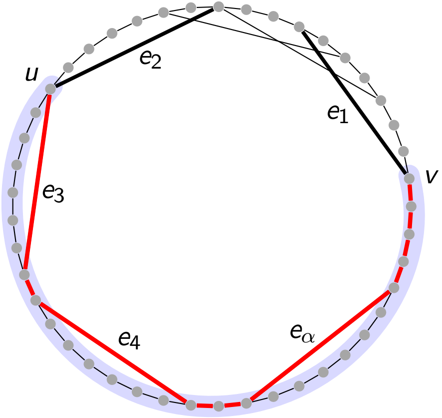
- Since $\rho \geq |F_1|$,

$$
\begin{aligned}
s(G) &\geq \frac{\alpha}{2}|F_1| + \frac{|F^\star|-1}{2} \\
&\geq \frac{\alpha}{2}\frac{q-|F^\star|}{\alpha} + \frac{|F^\star|-1}{2} \\
&\geq \frac{q-1}{2}
\end{aligned}
$$

Longer cycles





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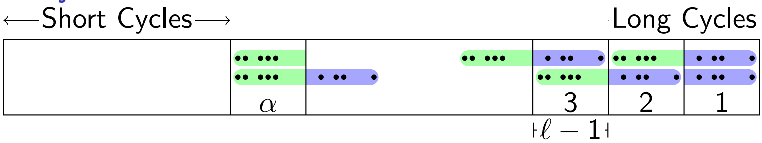

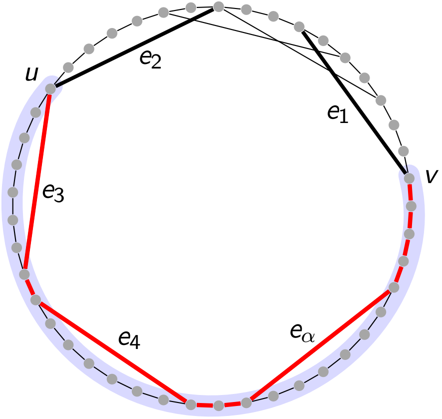

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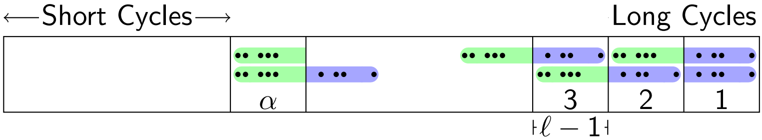
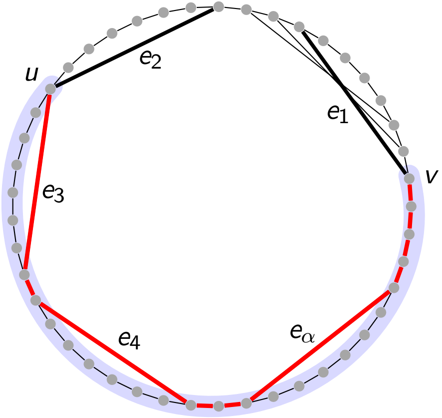
Longer cycles





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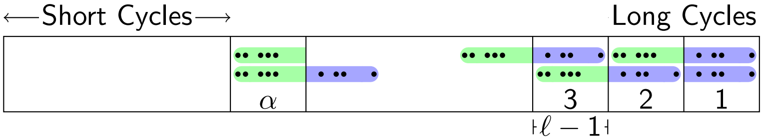

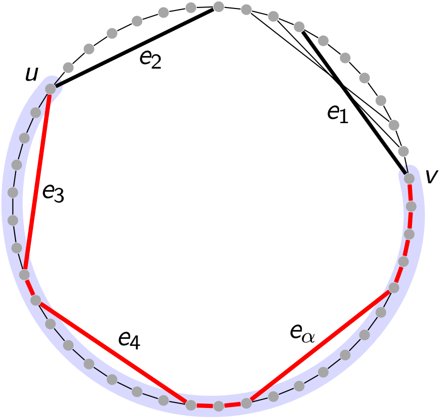

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- We exploit the structure in two cases to show

$$s(G) \geq \left(q - 1 - \frac{q}{\ell}\right)/2.$$

Summary and Open Problems

Theorem

If G is an n-vertex Hamiltonian graph with p chords, then $s(G) \geq \sqrt{p} - \frac{1}{2} \ln p - 1$.

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Thank You