

# Ordered Ramsey Theory and Track Representations of Graphs

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## Abstract

We study an ordered version of hypergraph Ramsey numbers using linearly ordered vertex sets, due to Fox, Pach, Sudakov, and Suk. In the  $k$ -uniform ordered path, the edges are the sets of  $k$  consecutive vertices in a linear vertex order. Moshkovitz and Shapira described its ordered Ramsey number as the solution to an enumeration problem involving higher-order multidimensional integer partitions. We give a short proof of an equivalent result: the Ramsey number is 1 plus the number of elements in the poset obtained by starting with a certain disjoint union of chains and repeatedly taking the poset of down-sets,  $k - 1$  times. We apply the resulting bounds to study the minimum number of interval graphs whose union is the line graph of the  $n$ -vertex complete graph, proving the conjecture of Heldt, Knauer, and Ueckerdt that this number grows with  $n$ . In fact, the growth rate is between  $\Omega(\frac{\log \log n}{\log \log \log n})$  and  $O(\log \log n)$ .

## 1 Introduction

We study an ordered version of Ramsey numbers for hypergraphs. An *ordered hypergraph* is a hypergraph on a linearly ordered vertex set. An ordered hypergraph  $H$  occurs as a *subhypergraph* of an ordered hypergraph  $H'$  if some order-preserving injection from  $V(H)$  to  $V(H')$  also preserves edges. We then say that  $H'$  *contains* (a copy of)  $H$ . Two ordered hypergraphs are *isomorphic* if each is a subhypergraph of the other.

Ramsey's Theorem concerns edge-colorings of ordinary (unordered) hypergraphs. Let  $K_r^k$  denote the complete  $k$ -uniform hypergraph with  $r$  vertices. For  $t, k, r \in \mathbb{N}$ , Ramsey's Theorem [18] implies that when  $n$  is sufficiently large, every  $t$ -coloring of  $E(K_n^k)$  contains a monochromatic copy of  $K_r^k$  (and hence also of every subhypergraph of  $K_r^k$ ). Because any complete ordered hypergraphs with the same number of vertices are isomorphic, the same statement holds in the ordered sense, including the part about subhypergraphs. This permits an ordered version of Ramsey numbers.

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**Definition 1.1.** Let  $H_1, \dots, H_t$  be ordered  $k$ -uniform hypergraphs, where  $k \geq 2$ . The  $t$ -color ordered Ramsey number  $\text{OR}(H_1, \dots, H_t)$  is the least integer  $n$  such that in every  $t$ -coloring of the edges of the ordered hypergraph  $K_n^k$  there is a copy of  $H_i$  in color  $i$  for some  $i$ . When  $H_1 = \dots = H_t = H$ , we abbreviate the notation to  $\text{OR}_t(H)$ ; this is the *diagonal case*. For a  $k$ -uniform ordered hypergraph  $H$ , an edge-coloring of the ordered hypergraph  $K_n^k$  is  $H$ -avoiding if it has no monochromatic  $H$ . The  $k$ -uniform ordered path  $P_r^k$  is the ordered hypergraph with  $r$  vertices in which the edges are the intervals of  $k$  consecutive vertices in the vertex ordering. Let  $m$  be the number of edges in  $P_r^k$ , so  $m = r - k + 1$ .

Note that  $\text{OR}_t(K_p^k)$  equals the classical  $t$ -color (unordered) Ramsey number for  $K_p^k$ . Although  $\text{OR}(H_1, \dots, H_t)$  is defined for general ordered  $k$ -uniform hypergraphs  $H_1, \dots, H_t$ , the problem is particularly fundamental for ordered paths, because this case has applications (see [8]) and because this case reduces to computing the size of a natural poset. (Choudum and Ponnusamy [3] used the term “ordered Ramsey number” for a different concept involving directed graphs in colored tournaments.)

Focusing on ordered paths, Fox, Pach, Sudakov, and Suk [8] defined  $N_k(t, m)$  to be the least integer  $N$  such that in every  $t$ -coloring of the  $k$ -subsets of  $N$  linearly ordered elements there is a monochromatic  $k$ -uniform ordered path with  $m$  edges (that is,  $N_k(t, m) = \text{OR}_t(P_{m+k-1}^k)$  in our notation). They proved  $2^{(m/t)^{t-1}} \leq N_3(t, m) \leq 2^{m^{t-1} \lg m}$ .

Eliáš and Matoušek [6] asked more generally for the growth rate of  $N_k(t, m)$ . Moshkovitz and Shapira [17] answered with upper and lower bounds that are towers of the same height. For a nonnegative integer  $h$ , define  $\text{tow}_h(x)$  by

$$\text{tow}_h(x) = \begin{cases} 2^{\text{tow}_{h-1}(x)} & h \geq 1 \\ x & h = 0 \end{cases}.$$

The subscript in our  $\text{tow}_h(x)$  is the number of 2s in the tower; thus  $\text{tow}_1(x)$  is exponential in  $x$  and  $\text{tow}_2(x)$  is “doubly-exponential” in  $x$ . In our notation, their bounds are as follows:

**Theorem 1.2** (Moshkovitz and Shapira [17]). *For  $t \geq 2$ ,  $k \geq 3$ , and  $m = r - k + 1 \geq 2$ ,*

$$\text{tow}_{k-2}(m^{t-1}/2\sqrt{t}) \leq \text{OR}_t(P_r^k) \leq \text{tow}_{k-2}(2m^{t-1}).$$

For  $k = 3$  they gave a sharper lower bound, yielding  $2^{\frac{2}{3}m^{t-1}/\sqrt{t}} \leq \text{OR}_t(P_r^3) \leq 2^{2m^{t-1}}$ .

Their bounds arose from their characterization of  $\text{OR}_t(P_r^k)$  using an enumerative problem. We give an equivalent characterization via a shorter and more direct proof. In order to state the result, we need some poset terminology. We view a partially ordered set or *poset* as a set equipped with an order relation.

Let  $Q$  be a poset. A *chain* in  $Q$  is a set of pairwise comparable elements, and an *antichain* is a set of pairwise incomparable elements. A *down-set* in  $Q$  a set  $S \subseteq Q$  such that if  $y \in S$  and  $x \leq y$ , then  $x \in S$ . Let  $|Q|$  denote the number of elements in  $Q$ . It is traditional to use  $J(Q)$  to denote the containment poset on the set of down-sets of  $Q$ . Let  $J^k(Q)$  denote the poset obtained from  $Q$  by repeatedly taking the poset of down-sets,  $k$  times. That is,  $J^0(Q) = Q$  and  $J^k(Q) = J(J^{k-1}(Q))$  for  $k \geq 1$ .

We phrase the enumerative statement for the general (non-diagonal) case because it is no more difficult than that of identical ordered paths. Moshkovitz and Shapira [17] stated essentially the same result in different language for the diagonal case, commenting that their argument extends to the general case. Hence it is proper to view this as their result, with a new proof here.

**Theorem 1.3.** *Given  $k \in \mathbb{N}$ , let  $r_1, \dots, r_t$  be positive integers greater than  $k$ . Let  $m_i = r_i - k + 1$  (thus  $m_i = |E(P_{r_i}^k)|$ ) and let  $Q$  be the poset consisting of disjoint chains with sizes  $m_1 - 1, \dots, m_t - 1$ . The ordered Ramsey number satisfies  $\text{OR}_t(P_{r_1}^k, \dots, P_{r_t}^k) = |J^{k-1}(Q)| + 1$ .*

A down-set in this poset  $Q$  is formed by taking some nonnegative number of elements from the bottom of each chain in  $Q$ ; hence  $J(Q)$  is the product of chains of sizes  $m_1, \dots, m_t$ . When  $t = 2$ , a down-set in  $J(Q)$  can be viewed as an integer partition. For general  $t$  and  $k$ , Moshkovitz and Shapira discussed the problem using higher-dimensional “line partitions”, where line partitions are higher-order analogues of integer partitions. They employed bijections to lattices of down-sets of appropriate posets. Nevertheless, the result is easier to state and prove directly in terms of down-sets as given above. The inductive nature of  $J^{k-1}(Q)$  yields recurrences for upper and lower bounds on  $\text{OR}_t(P_r^k)$  as immediate corollaries, which in turn yield the bounds in [17].

The resulting lower bound on  $\text{OR}_t(P_r^k)$  applies to the problem that motivated our study. In Section 3, we prove a conjecture of Heldt, Knauer, and Ueckerdt [12] about geometric representations of graphs. A  $t$ -interval representation of a graph  $G$  assigns each vertex the union of at most  $t$  intervals on the real line so that vertices are adjacent if and only if their assigned sets intersect. The interval number  $i(G)$  is the least  $t$  such that  $G$  has a  $t$ -interval representation. When  $i(G) = 1$ , we say that  $G$  is an interval graph and that the assignment of intervals is an interval representation.

A  $t$ -track representation of  $G$  is an expression of  $G$  as the union of  $t$  interval graphs. The track number  $\tau(G)$  (apparently introduced in [11]) is the least  $t$  such that  $G$  has a  $t$ -track representation. Note that a  $t$ -track representation yields a  $t$ -interval representation by placing interval representations of the  $t$  tracks in disjoint segments of the real line; thus  $i(G) \leq \tau(G)$ . (Note that  $\tau(G)$  is at most the number of caterpillars needed to decompose  $G$ ; see also [9, 10, 11, 13, 16].)

When  $i(G) = 1$ , also  $\tau(G) = 1$ , since such graphs are interval graphs. On the other hand,  $i(K_{5,3}) = 2 < \tau(K_{5,3})$ . Heldt, Knauer, and Ueckerdt [12] conjectured that the track number is not bounded by any function of the interval number, even when restricted to line graphs. Letting  $L(G)$  denote the line graph of a graph  $G$ , always  $i(L(G)) \leq 2$ , since each edge of  $G$  is incident to two vertices, and the edges incident to a common vertex can be given intervals having a common point. Since track representations are hereditary over induced subgraphs, the track number is unbounded for line graphs if and only if it is unbounded for  $L(K_n)$ .

To study  $\tau(L(K_n))$ , we relate  $t$ -color ordered Ramsey numbers of particular 3-uniform ordered hypergraphs to  $t$ -track representations of  $L(K_n)$ . Letting  $t$  be the track number of  $L(K_n)$ , we prove

$$\text{OR}_{t-3}(P_4^3) \leq n < \text{OR}_t(P') \leq \text{OR}_t(K_6^3),$$

where  $P'$  is obtained from  $P_6^3$  by adding the two edges  $\{1, 2, 5\}$  and  $\{2, 5, 6\}$ . By inverting the relationship between  $t$  and  $n$ , we obtain

$$\Omega\left(\frac{\lg \lg n}{\lg \lg \lg n}\right) \leq \tau(L(K_n)) \leq O(\lg \lg n).$$

## 2 Ordered Ramsey Numbers for $k$ -Uniform Paths

Let  $[r] = \{1, \dots, r\}$ . We will use  $[r]$  as the vertex set of  $P_r^k$ , so the  $r - k + 1$  edges are the sets of  $k$  consecutive elements of  $[r]$ . Note that  $P_r^2$  is the ordinary  $r$ -vertex path with vertices indexed in order. To clarify the role of the vertex ordering, consider instead the 3-vertex path with edges  $\{1, 2\}$  and  $\{1, 3\}$ . Avoiding the target when  $t$ -coloring  $E(K_n)$  requires the edges leaving vertex 1 to have distinct colors, so  $n - 1 \leq t$ . Coloring edges by their higher endpoint achieves the bound, so the ordered Ramsey number equals  $t + 2$ . For the monotone path  $P_3^2$ , Theorem 1.3 with  $r_1 = \dots = r_t = 3$  and  $k = 2$  yields the much higher value  $2^t + 1$ .

For  $k = 2$  in general, Theorem 1.3 states  $\text{OR}_t(P_{m_1+1}^2, \dots, P_{m_t+1}^2) = 1 + \prod_{i=1}^t m_i$ . This case is special enough that it also arose in the tournament model of Choudum and Ponnusamy [3] (stated without proof). It is a general form of the Erdős–Szekeres Theorem, as observed by Seidenberg [19] and by Moshkovitz and Shapira [17]. Its proof provides motivation for the general argument. To show the upper bound, let  $E(K_n)$  be  $t$ -colored with no increasing  $P_{r_i+1}$  in color  $i$ , and compute for  $x \in [n]$  the  $t$ -tuple of orders of longest paths reaching  $x$  in the various colors; only  $\prod_{i=1}^t r_i$  such  $t$ -tuples are available. To achieve equality, view  $[n]$  as  $\prod_{i=1}^t [r_i]$  and let the color of an edge be the first index where the endpoints differ.

In the proof of the general upper bound, we will need the smallest down-set containing a given set of elements in a poset  $Q$ . For  $S \subseteq Q$ , the *down-set generated by  $S$* , denoted  $D(S)$ , is  $\{u \in Q : u \leq v \text{ for some } v \in S\}$ . We restate Theorem 1.3 in a slightly different way to simplify notation in the proof.

**Theorem 2.1.** *Given  $k \in \mathbb{N}$ , let  $r_1, \dots, r_t$  be positive integers greater than  $k$ . Let  $Q$  be the poset consisting of disjoint chains with sizes  $r_1 - k, \dots, r_t - k$ . Letting  $Q_1 = Q$  and  $Q_i = J(Q_{i-1})$  for  $i > 1$ , the ordered Ramsey number satisfies  $\text{OR}_t(P_{r_1}^k, \dots, P_{r_t}^k) = |Q_k| + 1$ .*

*Proof.* The case  $k = 1$  is just the pigeonhole principle; consider  $k \geq 2$ .

*Lower Bound.* For  $n = |Q_k|$ , we will construct a  $t$ -edge-coloring of  $K_n^k$  that avoids  $P_{r_i}^k$  in color  $i$  for each  $i$ . First consider  $x, y \in Q_j$ , where  $2 \leq j \leq k$ . If  $x \not\preceq y$ , then  $x$  does not contain  $y$  when they are viewed as down-sets in  $Q_{j-1}$ ; let  $f(x, y)$  be a particular element of the set  $y - x$  in  $Q_{j-1}$ . For  $x, y, z \in Q_j$ , if  $x \not\preceq y$  and  $y \not\preceq z$ , then  $y$  (in  $Q_{j-1}$ ) contains  $f(x, y)$  but not  $f(y, z)$ . Since  $y$  is a down-set in  $Q_{j-1}$ , we obtain  $f(x, y) \not\preceq f(y, z)$  in  $Q_{j-1}$ .

A list  $x_1, \dots, x_s$  in a poset is *descent-free* if  $x_i \not\preceq x_{i+1}$  for  $1 \leq i \leq s - 1$ . We extend  $f$  to descent-free lists in  $Q_j$  by setting  $f(x_1, \dots, x_s) = (f(x_1, x_2), \dots, f(x_{s-1}, x_s))$ . The image of a descent-free  $s$ -list in  $Q_j$  is a descent-free  $(s - 1)$ -list in  $Q_{j-1}$ . Let  $f^0$  be the identity map. For a descent-free  $s$ -list  $x_1, \dots, x_s$  in  $Q_j$ , where  $s, j > d > 1$ , define  $f^d(x_1, \dots, x_s) = f(f^{d-1}(x_1, \dots, x_s))$ . When defined,  $f^d(x_1, \dots, x_s)$  is a descent-free  $(s - d)$ -list in  $Q_{j-d}$ .

Now let  $y_1, \dots, y_n$  be a linear extension of  $Q_k$ ; that is, an ordering of  $Q_k$  such that  $y_i \leq y_j$  implies  $i \leq j$ . Each sublist of a linear extension is descent-free. Also, for sublist  $x_1, \dots, x_k$ , note that  $f^{k-1}(x_1, \dots, x_k)$  is a single element in  $Q$ . Color  $\{x_1, \dots, x_k\}$  (as an edge in  $K_n^k$ ) with the index of the chain in  $Q$  that contains  $f^{k-1}(x_1, \dots, x_k)$ . If  $x_1, \dots, x_s$  is a sublist of  $y_1, \dots, y_n$  whose consecutive  $k$ -lists all have color  $i$ , then  $f^{k-1}(x_1, \dots, x_s)$  is a descent-free  $(s - k + 1)$ -list in the  $i$ th chain of  $Q$ . A descent-free list in a chain is strictly increasing, so  $s - k + 1 \leq r_i - k$ . Thus  $s < r_i$ , and hence the coloring avoids  $P_{r_i}^k$  in color  $i$  for each  $i$ .

*Upper bound.* Given a  $t$ -edge-coloring  $\phi$  of  $E(K_n^k)$  with vertex set  $[n]$  that avoids  $P_{r_i}^k$  in color  $i$  for each  $i$ , it suffices to define an injection from  $[n]$  to  $Q_k$ . View each vertex subset  $Y \subseteq [n]$  as an increasing list, with  $Y^- = Y - \max Y$  and  $Y^+ = Y - \min Y$ .

For  $1 \leq j \leq k < n$ , we construct  $g_j: \binom{[n]}{j} \rightarrow Q_{k-j+1}$  such that

$$g_j(Y^-) \not\leq g_j(Y^+) \text{ in } Q_{k-j+1} \text{ when } Y \in \binom{[n]}{j+1}. \quad (*)$$

This suffices, since  $g_1$  will then be the desired injection.

We first define  $g_k$ . For  $X \in \binom{[n]}{k}$ , let  $i = \phi(X)$ , and let  $w_1, \dots, w_{r_i-k}$  be the  $i$ th chain in  $Q$ . Set  $g_k(X) = w_h$ , where  $h$  is the largest integer such that some  $i$ -colored copy of  $P_{h+k-1}^k$  has last edge  $X$ . Note that  $h \leq r_i - k$ , since  $\phi$  yields no  $i$ -colored copy of  $P_{r_i}^k$ . If  $\phi(Y^-) = \phi(Y^+)$  for some  $Y \in \binom{[n]}{k+1}$ , then  $g_k(Y^+) > g_k(Y^-)$ , and otherwise  $g_k(Y^+)$  and  $g_k(Y^-)$  are incomparable in  $Q$ . In either case,  $g_k(Y^-) \not\leq g_k(Y^+)$ .

Now consider smaller  $j$ , with  $g_{j+1}$  already defined and satisfying  $(*)$ . For  $X \in \binom{[n]}{j}$ , let  $g_j(X) = D(\{g_{j+1}(Y) : Y^+ = X\})$ . Since  $Q_{k-j+1} = J(Q_{k-j})$ , by construction  $g_j(X) \in Q_{k-j+1}$ . Now consider  $Y \in \binom{[n]}{j+1}$ ; not  $g_{j+1}(Y) \in g_j(Y^+)$ . If also  $g_{j+1}(Y) \in g_j(Y^-)$ , then since  $g_j(Y^-)$  is a down-set there exists  $Z \in \binom{[n]}{j+1}$  such that  $g_{j+1}(Y) \leq g_{j+1}(Z)$  in  $Q_{k-j}$  and  $Z^+ = Y^-$ . That is,  $g_{j+1}(W^-) \geq g_{j+1}(W^+)$  for  $W = Y \cup Z$ , which contradicts  $(*)$  for  $g_{j+1}$ . Thus  $g_{j+1}(Y) \in g_j(Y^+) - g_j(Y^-)$ , which yields  $g_j(Y^-) \not\leq g_j(Y^+)$ . ■

With  $Q$  being the disjoint union of chains of sizes  $m_1 - 1, \dots, m_t - 1$ , we have noted that  $|Q_1| = \sum(m_i - 1)$  and  $|Q_2| = \prod m_i$ , since  $Q_2$  is the product of chains of sizes  $m_1, \dots, m_t$ .

For  $k \geq 3$ , exact values are unknown. Since down-sets correspond bijectively to antichains,  $|Q_3|$  is the number of antichains in the chain-product. As noted by Moshkovitz and Shapira, this is already a famous problem when all  $m_i$  equal 2: counting the antichains among subsets of  $[t]$  is known as *Dedekind's Problem*. The family of all subsets of  $\binom{[t]}{\lfloor t/2 \rfloor}$  yields the lower bound  $2^{\binom{t}{\lfloor t/2 \rfloor}}$ ; the family of all subsets of  $Q_2$  yields the upper bound  $2^{2^t}$ . Not surprisingly, the lower bound is sharper: Kleitman and Markovsky [15] proved

$$\binom{t}{\lfloor t/2 \rfloor} \leq \lg |Q_3| \leq \binom{t}{\lfloor t/2 \rfloor} \left(1 + O\left(\frac{\log n}{n}\right)\right).$$

The upper bound was further refined by Korshunov [14]. Note that  $\binom{t}{\lfloor t/2 \rfloor} \approx 2^t / \sqrt{2\pi t}$  and  $3 > \sqrt{2\pi}$  yield the lower bound on  $\text{OR}_t(P_4^3)$  given by Moshkovitz and Shapira.

Asymptotics are also known for the size of the largest rank in  $Q_2$  when  $m_1 = \dots = m_t = m > 2$ . As long as  $m \in o(e^t/\sqrt{t})$ , the size is asymptotic to  $m^{t-1}/\sqrt{\pi t/6}$  (see [2]). Since

$\sqrt{\pi/6} < 2$ , for  $t > \log m + \frac{1}{2} \log \log m$  this slightly improves the lower bound on  $\text{OR}_t(P_{m+2}^3)$  from Moshkovitz and Shapira [17] (stated in Theorem 1.2).

For larger  $k$ , we iterate the argument. Our point is that this is how the enumerative characterization yields the bounds of [17]. Using the constant 2 instead of  $\sqrt{\pi/6}$  allows in Theorem 1.2 a uniform lower bound for all  $t$  without reference to the logarithmic term in the argument to the tower function.

**Corollary 2.2.** *Let  $Q$  be the disjoint union of  $t$  chains of size  $m - 1$ . Define  $Q_k$  for  $k \geq 1$  as in Theorem 2.1. For  $k \geq 3$  and  $t > \log m + \frac{1}{2} \log \log m$ ,*

$$\text{tow}_{k-2}(m^{t-1}/\sqrt{\pi t/6} - O(t^2)) \leq |Q_k| \leq \text{tow}_{k-2}(m^t)$$

*Proof.* The upper bound arises from  $|Q_2| = m^t$  by iteratively taking all subsets.

For the lower bound, we define an analogue of the tower function using the middle binomial coefficient. Let  $b_0(x) = x$ . For  $h \geq 1$ , let  $b_h(x) = \binom{b_{h-1}(x)}{\lfloor b_{h-1}(x)/2 \rfloor}$ . In terms of the tower function, Stirling's approximation yields  $b_h(x) \geq \text{tow}_h(x - O(\lg x))$ .

Let  $a_k$  be the maximum size of a family in  $Q_k$  consisting of down-sets in  $Q_{k-1}$  that have the same size; these sets form an antichain in  $Q_k$ . Thus  $|Q_{k+1}| \geq 2^{a_k}$ , but also  $a_{k+1} \geq \binom{a_k}{\lfloor a_k/2 \rfloor}$ . Thus  $a_k \geq b_{k-2}(a_2)$ . With  $a_2 = m^{t-1}/\sqrt{\pi t/6}$ , we have

$$|Q_k| \geq 2^{a_{k-1}} \geq 2^{b_{k-3}(a_2)} \geq \text{tow}_{k-2}(m^{t-1}/\sqrt{\pi t/6} - O(t^2)). \quad \blacksquare$$

Since we do not have an exact computation of  $|J^{k-1}(Q)|$ , recursive bounds may be of interest. In an earlier version of this paper, methods like those of Theorem 2.1 were used to give short proofs of several such bounds.

**Theorem 2.3.** *For  $m = r - k + 1$  and  $r > k \geq 3$ , the following recursive bounds hold:*

$$\begin{aligned} \text{OR}_t(P_{r+1}^k) &\geq \text{OR}_{\binom{t}{\lfloor t/2 \rfloor}}(P_r^{k-1}) & \text{OR}_t(P_{k+1}^k) &\leq \text{OR}_{2^t}(P_k^{k-1}) \\ \text{OR}_t(P_r^k) &\leq \text{OR}_{m^t}(P_k^{k-1}) & \text{OR}_t(P_{k+2}^k) &\leq \text{OR}_{2^{2t}}(P_k^{k-1}). \end{aligned}$$

It would be interesting to obtain good bounds on other ordered Ramsey numbers. For example, an ordered  $k$ -uniform version of a cycle can be defined by letting the edges of  $C_r^k$  with vertex set  $[r]$  be all sets of  $k$  members of  $[r]$  whose congruence classes modulo  $r$  are consecutive. Is  $\text{OR}_t(C_r^k)$  much larger than  $\text{OR}_t(P_r^k)$ ? Also, the ordinary Ramsey number of a graph is bounded by the ordered Ramsey numbers of all its orderings. Given an unordered graph, which vertex orderings lead to the smallest and largest ordered Ramsey numbers? For steps in these directions, see Balko and Král [1]. Another paper studying special cases of ordered Ramsey numbers for graphs (2-uniform hypergraphs) is [4].

### 3 Separating Interval Number and Track Number

Recall from the introduction the definitions of  $t$ -interval representation,  $t$ -track representation, interval number  $i(G)$ , and track number  $\tau(G)$  for a graph  $G$ . We observed that always  $i(G) \leq \tau(G)$ . Heldt, Knauer, and Ueckerdt [12] conjectured that  $\tau(G)$  can grow while  $i(G) = 2$  even for the line graphs of complete graphs. We prove this in the next theorem, using ordered Ramsey numbers. First we pose a stronger conjecture:

**Conjecture 3.1.** *For a sequence  $(G_n)_{n=1}^\infty$  of graphs, if  $\chi(G_n) \rightarrow \infty$ , then  $\tau(L(G_n)) \rightarrow \infty$ .*

**Theorem 3.2.** *Let  $P'$  be the ordered hypergraph obtained from the ordered  $P_6^3$  with vertices  $\{1, \dots, 6\}$  by adding the edges  $\{1, 2, 5\}$  and  $\{2, 5, 6\}$ . If  $n \geq \text{OR}_t(P')$ , then  $\tau(L(K_n)) > t$ .*

*Proof.* From a  $t$ -track representation of  $L(K_n)$ , we define a  $t$ -coloring  $\phi$  of  $E(K_n^3)$ . For  $x, y, z \in [n]$  with  $x < y < z$ , let  $\phi(x, y, z)$  be the index of a track where the intervals for  $xy$  and  $yz$  intersect. Since  $xy$  and  $yz$  are incident in  $K_n$ ,  $\phi$  is well-defined. The intervals for disjoint pairs do not overlap in any track, since they are not adjacent in  $L(K_n)$ .

Since  $n \geq \text{OR}_t(P')$ , under  $\phi$  there is a monochromatic ordered copy of  $P'$ ; let its vertices be  $\{x_1, \dots, x_6\}$  in order, and let its edges have color  $s$ . The intervals for  $x_i x_{i+1}$  and  $x_j x_{j+1}$  in track  $s$  overlap when  $|i - j| = 1$ , but not when  $|i - j| > 1$ . Hence we have a path of five successively overlapping intervals. Since this copy of  $P'$  is monochromatic, the interval for  $x_2 x_5$  must intersect the intervals for  $x_1 x_2$  and  $x_5 x_6$ , but it must not intersect the interval for  $x_3 x_4$  that lies between them. Hence these incidences cannot be represented in a single track, and  $L(K_n)$  has no  $t$ -track representation. ■

For classical Ramsey numbers, Conlon, Fox, and Sudakov [5] proved  $R_t(K_6^3) \leq 2^{2^{(4+o(1))t \lg t}}$ .

**Corollary 3.3.**  $\tau(L(K_n)) \geq \frac{\lg \lg n}{(4+o(1)) \lg \lg n}$ .

*Proof.* We now have  $\text{OR}_t(P') \leq \text{OR}_t(K_6^3) = R_t(K_6^3) \leq 2^{2^{(4+o(1))t \lg t}}$ . If  $n$  is larger than this bound, then  $n > \text{OR}_t(P')$  and  $\tau(L(K_n)) > t$ . To express the lower bound in terms of  $n$ , we seek  $t$  such that  $n \geq 2^{2^{(4+o(1))t \lg t}}$ , yielding the claimed bound. ■

To obtain an upper bound on  $\tau(L(K_n))$ , we again use ordered Ramsey theory.

**Theorem 3.4.** *If  $n < \text{OR}_t(P_4^3)$ , then  $\tau(L(K_n)) \leq t + 2$ .*

*Proof.* We convert a  $P_4^3$ -avoiding  $t$ -coloring  $\phi$  of  $E(K_n^3)$  into a representation of  $L(K_n)$  with  $t + 2$  tracks. If  $\phi(\{x, y, z\}) = i$ , where  $x < y < z$ , then call  $xy$  a *left pair* and  $yz$  a *right pair* for color  $i$ . Since  $\phi$  avoids  $P_4^3$ , no pair  $xy$  is a left and right pair for the same color.

For  $xy \in E(K_n)$  with  $x < y$ , define the interval for  $xy$  in track  $i$  as follows, where  $\epsilon = 1/3$ :

1. If  $xy$  is a left pair for  $i$ , then use the interval  $[y - \epsilon, y + \epsilon]$ .
2. If  $xy$  is a right pair for  $i$ , then use the interval  $[x - \epsilon, x + \epsilon]$ .
3. If  $xy$  is not a left or right pair for  $i$ , then let the interval intersect no others.

An edge in  $L(K_n)$  involves three vertices of  $K_n$ , say  $\{x, y, z\}$  with  $x < y < z$ . The edge  $\{xy, yz\}$  is represented in the track whose index is  $\phi(\{x, y, z\})$ . The edge  $\{xy, xz\}$  is represented in the “left track”, where each pair with lower endpoint  $x$  gets the interval  $[x - \varepsilon, x + \varepsilon]$ . The edge  $\{xz, yz\}$  is represented in the “right track”, where each pair with higher endpoint  $z$  gets the interval  $[z - \varepsilon, z + \varepsilon]$ . ■

Corollary 3.3 and the lower bound on  $\text{OR}_t(P_4^3)$  (Theorem 1.2) together yield the following:

**Corollary 3.5.**  $\Omega\left(\frac{\lg \lg n}{\lg \lg \lg n}\right) \leq \tau(L(K_n)) \leq O(\lg \lg n)$ .

Improvement of the asymptotic upper bound on  $\text{OR}_t(P')$  would lead to an improvement of the asymptotic lower bound on  $\tau(L(K_n))$ , the aim being to eliminate the denominator in the lower bound. For  $R_t(K_6^3)$ , the known upper bound is  $2^{2^{(4+o(1))t \lg t}}$ , but the known lower bound is only  $2^{2^{ct}}$ , so the improvement could happen there. Since  $P'$  is a proper subhypergraph, improvement is possible without determining  $R_t(K_6^3)$ . Since  $P'$  consists of two 3-uniform ordered paths,  $P_6^3$  and  $P_4^3$ , whose first two vertices and last two vertices are identified, finding stronger upper bounds on  $\text{OR}_t(P_6^3)$  and  $\text{OR}_t(P_4^3)$  may suffice.

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