Monotone Paths in Dense Edge-Ordered Graphs

Kevin G. Milans (milans@math.wvu.edu)

West Virginia University

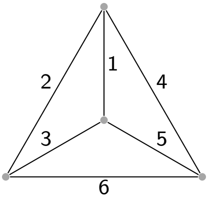
AMS Spring Southeastern Sectional Meeting
University of Georgia
Athens, GA
March 5, 2016

Monotone paths

- Let G be a graph whose edges are ordered according to a labeling φ.

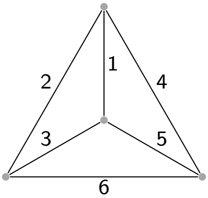
Monotone paths

- Let G be a graph whose edges are ordered according to a labeling φ.



Monotone paths

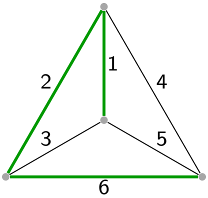
- Let G be a graph whose edges are ordered according to a labeling φ.



- A monotone path traverses edges in increasing order.

Monotone paths

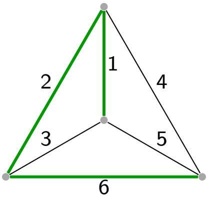
- Let G be a graph whose edges are ordered according to a labeling φ.



- A monotone path traverses edges in increasing order.

Monotone paths

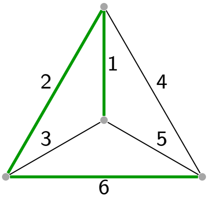
- Let G be a graph whose edges are ordered according to a labeling φ.



- A monotone path traverses edges in increasing order.
- The altitude of G, denoted $f(G)$, is the maximum integer k such that every edge-ordering of G has a monotone path of length k.

Monotone paths

- Let G be a graph whose edges are ordered according to a labeling φ.



- A monotone path traverses edges in increasing order.
- The altitude of G, denoted $f(G)$, is the maximum integer k such that every edge-ordering of G has a monotone path of length k.
- [Chvátal–Komlós (1971)] What is $f(K_n)$?

Prior work

Theorem (Graham–Kleitman (1973))

$\sqrt{n - \frac{3}{4}} - \frac{1}{2} \leq f(K_n) \leq \frac{3n}{4}$

Prior work

Theorem (Graham–Kleitman (1973))

$\sqrt{n - \frac{3}{4}} - \frac{1}{2} \leq f(K_n) \leq \frac{3n}{4}$

- Rödl: Graham–Kleitman and design theory give $f(K_n) \leq (\frac{2}{3} + o(1))n$

Prior work

Theorem (Graham–Kleitman (1973))

$$\sqrt{n - \frac{3}{4}} - \frac{1}{2} \leq f(K_n) \leq \frac{3n}{4}$$

- Rödl: Graham–Kleitman and design theory give $f(K_n) \leq (\frac{2}{3} + o(1))n$
- Alspach–Heinrich–Graham (unpublished): $f(K_n) \leq (\frac{7}{12} + o(1))n$

Prior work

Theorem (Graham–Kleitman (1973))

$\sqrt{n - \frac{3}{4}} - \frac{1}{2} \leq f(K_n) \leq \frac{3n}{4}$

- Rödl: Graham–Kleitman and design theory give $f(K_n) \leq (\frac{2}{3} + o(1))n$
- Alspach–Heinrich–Graham (unpublished): $f(K_n) \leq (\frac{7}{12} + o(1))n$

Theorem (Calderbank–Chung–Sturtevant (1984))

$f(K_n) \leq (\frac{1}{2} + o(1))n$

Prior work II

- Roditty–Shoham–Yuster (2001): the max. altitude of a planar graph is in $\{5, 6, 7, 8, 9\}$.

Prior work II

- Roditty–Shoham–Yuster (2001): the max. altitude of a planar graph is in $\{5, 6, 7, 8, 9\}$.
- Alon (2003): the max. altitude of a k-regular graph is in $\{k, k+1\}$.

Prior work II

- Roditty–Shoham–Yuster (2001): the max. altitude of a planar graph is in $\{5, 6, 7, 8, 9\}$.
- Alon (2003): the max. altitude of a k-regular graph is in $\{k, k+1\}$.
- Mynhardt–Burger–Clark–Falvai–Henderson (2005): the max. altitude of a 3-regular graph is 4, achieved by the flower snarks.

Prior work II

- Roditty–Shoham–Yuster (2001): the max. altitude of a planar graph is in $\{5, 6, 7, 8, 9\}$.
- Alon (2003): the max. altitude of a k-regular graph is in $\{k, k+1\}$.
- Mynhardt–Burger–Clark–Falvai–Henderson (2005): the max. altitude of a 3-regular graph is 4, achieved by the flower snarks.

Theorem (De Silva–Molla–Pfender–Retter–Tait (2015+))

- $f(Q_n) \geq n / \lg n$

Prior work II

- Roditty–Shoham–Yuster (2001): the max. altitude of a planar graph is in $\{5, 6, 7, 8, 9\}$.
- Alon (2003): the max. altitude of a k-regular graph is in $\{k, k+1\}$.
- Mynhardt–Burger–Clark–Falvai–Henderson (2005): the max. altitude of a 3-regular graph is 4, achieved by the flower snarks.

Theorem (De Silva–Molla–Pfender–Retter–Tait (2015+))

- $f(Q_n) \geq n/\lg n$
- *If* $p(n) = \omega(\log n/\sqrt{n})$*, then* $f(G(n,p)) \geq (1-o(1))\sqrt{n}$ *with probability tending to* 1*.*

Random edge-orderings

Theorem (Lavrov–Loh (2015+))

- *With probability tending to* 1*, a random edge-labeling of* K_n *has a monotone path of length* $0.85n$*.*

Random edge-orderings

Theorem (Lavrov–Loh (2015+))

- *With probability tending to* 1*, a random edge-labeling of* K_n *has a monotone path of length* $0.85n$*.*
- *With probability at least* $1/e - o(1)$*, a random edge-labeling of* K_n *has a Hamiltonian monotone path.*

Random edge-orderings

Theorem (Lavrov–Loh (2015+))

- *With probability tending to* 1*, a random edge-labeling of* K_n *has a monotone path of length* $0.85n$*.*
- *With probability at least* $1/e - o(1)$*, a random edge-labeling of* K_n *has a Hamiltonian monotone path.*

Conjecture (Lavrov–Loh)

With high probability, a random edge-labeling of K_n has a Hamiltonian monotone path.

Our result

Theorem (Graham–Kleitman (1973))

$f(K_n) \geq \sqrt{n - \frac{3}{4}} - \frac{1}{2}$

Our result

Theorem (Graham–Kleitman (1973))
$f(K_n) \geq \sqrt{n - \frac{3}{4}} - \frac{1}{2}$

Theorem (Rödl (1973))
If G has average degree d, then $f(G) \geq (1 - o(1))\sqrt{d}$.

Our result

Theorem (Graham–Kleitman (1973))
$f(K_n) \geq \sqrt{n - \frac{3}{4}} - \frac{1}{2}$

Theorem (Rödl (1973))
If G has average degree d, then $f(G) \geq (1 - o(1))\sqrt{d}$.

Theorem
Let G be an n-vertex graph, and let $s = Cn^{1/3}(\lg n)^{2/3}$. If G has average degree d, then

$$f(G) \geq \frac{d}{4s}\left(1 - \frac{2}{d}\right)\left(1 - \frac{1}{s}\right)\left(1 - \frac{4s^2}{d-2}\right).$$

Our result

Theorem (Graham–Kleitman (1973))
$f(K_n) \geq \sqrt{n - \frac{3}{4}} - \frac{1}{2}$

Theorem (Rödl (1973))
If G has average degree d, then $f(G) \geq (1 - o(1))\sqrt{d}$.

Theorem
Let G be an n-vertex graph, and let $s = Cn^{1/3}(\lg n)^{2/3}$. If G has average degree d, then

$$f(G) \geq \frac{d}{4s}\left(1 - \frac{2}{d}\right)\left(1 - \frac{1}{s}\right)\left(1 - \frac{4s^2}{d-2}\right).$$

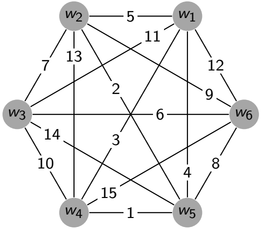
Corollary
$f(K_n) \geq (\frac{1}{20} - o(1))(n/\lg n)^{2/3}$

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

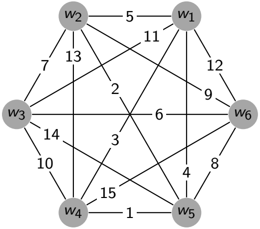
The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.



The height table

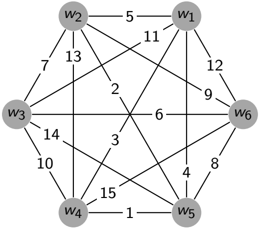
- Let G be a graph with vertices $w_1, \ldots, w_n$.



- The height table A has a column for each vertex in G.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

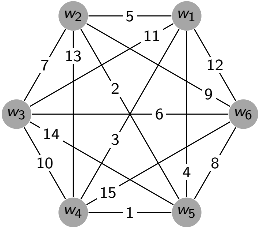


w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

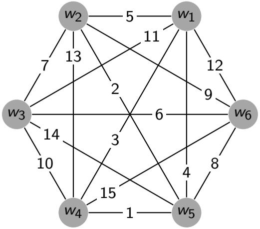


w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

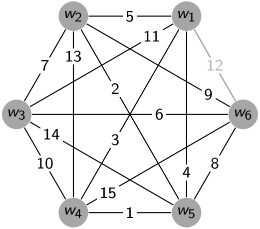


w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

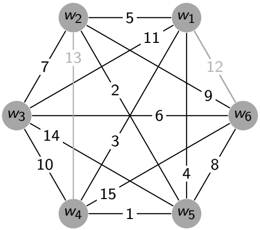


16					
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

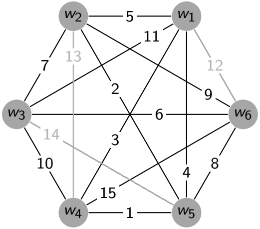


16	24				
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

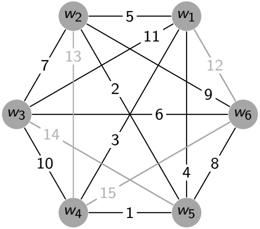


w_1	w_2	w_3	w_4	w_5	w_6
16	24	35			

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

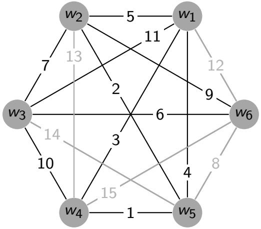


16	24	35	46		
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

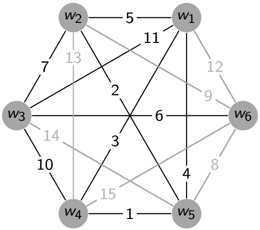


16	24	35	46	56	
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

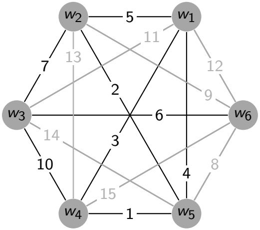


16	24	35	46	56	62
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

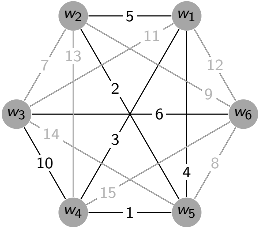


13					
16	24	35	46	56	62
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

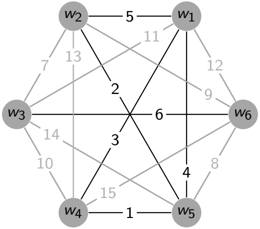


13	23				
16	24	35	46	56	62
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

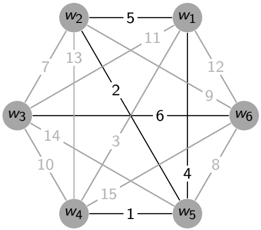


13	23	34			
16	24	35	46	56	62
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

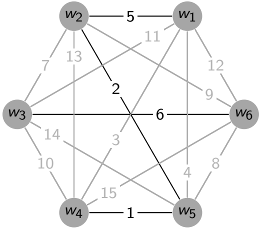


13	23	34	41		
16	24	35	46	56	62
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

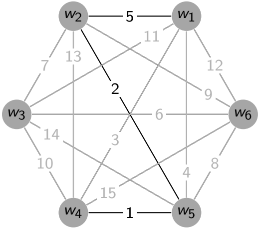


13	23	34	41	51	
16	24	35	46	56	62
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

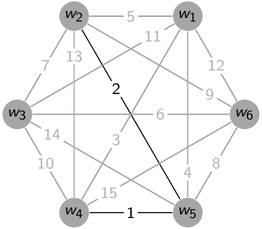


13	23	34	41	51	63
16	24	35	46	56	62
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

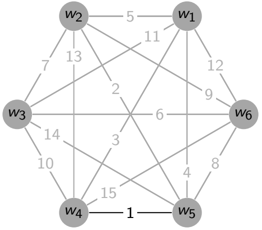


12					
13	23	34	41	51	63
16	24	35	46	56	62
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

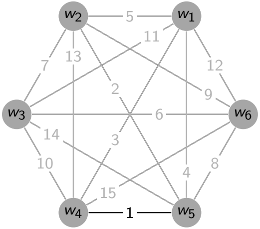


12	25				
13	23	34	41	51	63
16	24	35	46	56	62
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

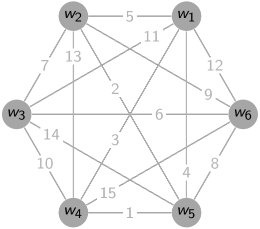


12	25	–			
13	23	34	41	51	63
16	24	35	46	56	62
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

- Let G be a graph with vertices $w_1, \ldots, w_n$.

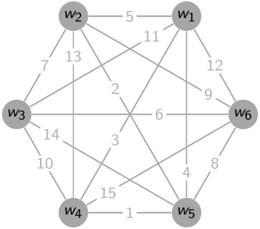


12	25	–	45		
13	23	34	41	51	63
16	24	35	46	56	62
w_1	w_2	w_3	w_4	w_5	w_6

- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.

The height table

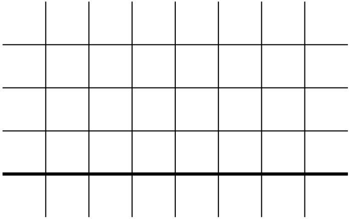
- Let G be a graph with vertices $w_1, \ldots, w_n$.



12	25	–	45		
13	23	34	41	51	63
16	24	35	46	56	62
w_1	w_2	w_3	w_4	w_5	w_6

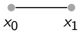
- The height table A has a column for each vertex in G.
- Fill in the cells row by row, from bottom to top.
- Next entry in column i is the edge incident to w_i with largest label not already appearing in A.
- The height of an edge e, denoted $h(e)$, is the index of the row containing e. For example, $h(w_1 w_2) = 3$.

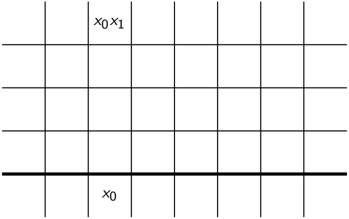
Monotone path extension



- Given G, construct the height table A.

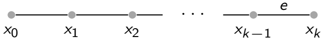
Monotone path extension

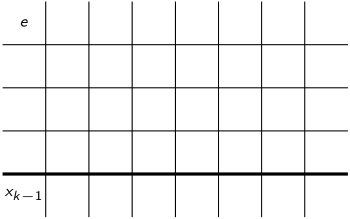




- Given G, construct the height table A.
- Let x_0x_1 be a max-height edge in column x_0. Set $P = x_0x_1$.

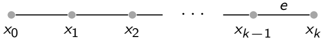
Monotone path extension

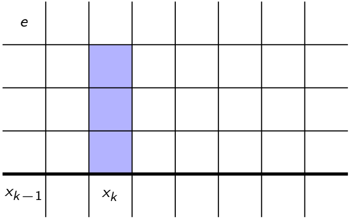




- Let P be a monotone path $x_0 \ldots x_k$; let $e = x_{k-1}x_k$. We extend P as follows.

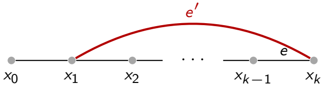
Monotone path extension

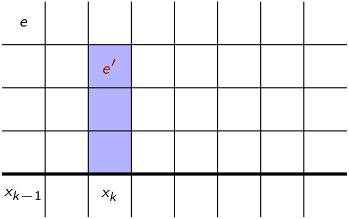




- Let P be a monotone path $x_0 \ldots x_k$; let $e = x_{k-1}x_k$. We extend P as follows.
  - Note $\varphi(e') > \varphi(e)$ if e' is in a lower row in column x_k.

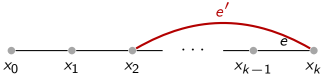
Monotone path extension

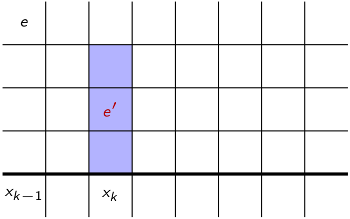




- Let P be a monotone path $x_0 \ldots x_k$; let $e = x_{k-1}x_k$. We extend P as follows.
  - Note $\varphi(e') > \varphi(e)$ if e' is in a lower row in column x_k.

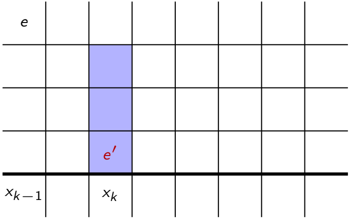
Monotone path extension





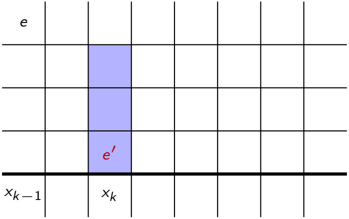
- Let P be a monotone path $x_0 \ldots x_k$; let $e = x_{k-1}x_k$. We extend P as follows.
  - Note $\varphi(e') > \varphi(e)$ if e' is in a lower row in column x_k.

Monotone path extension



- Let P be a monotone path $x_0 \ldots x_k$; let $e = x_{k-1}x_k$. We extend P as follows.
  - Note $\varphi(e') > \varphi(e)$ if e' is in a lower row in column x_k.
  - Let e' be the highest such edge joining x_k to a vertex outside $\{x_1, \ldots, x_{k-1}\}$.
  - Extend P along e'.

Monotone path extension

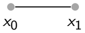


- Let P be a monotone path $x_0 \dots x_k$; let $e = x_{k-1}x_k$. We extend P as follows.
  - Note $\varphi(e') > \varphi(e)$ if e' is in a lower row in column x_k.
  - Let e' be the highest such edge joining x_k to a vertex outside $\{x_1, \dots, x_{k-1}\}$.
  - Extend P along e'.
- Iteratively extending gives $f(G) \geq \left\lfloor 1/2 + \sqrt{d} \right\rfloor$, matching Rödl's bound asymptotically.

The algorithm

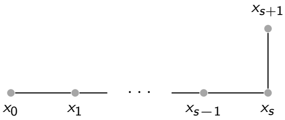
- Given G, construct the height table A. Let $P = x_0x_1$, where x_0x_1 is a max-height edge.

The algorithm



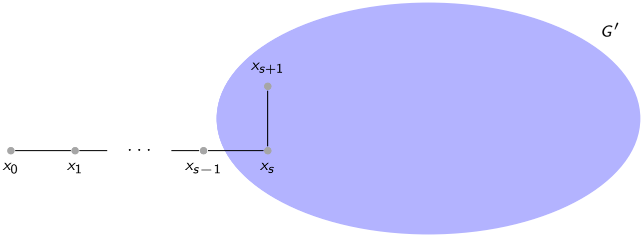
- Given G, construct the height table A. Let $P = x_0x_1$, where x_0x_1 is a max-height edge.

The algorithm



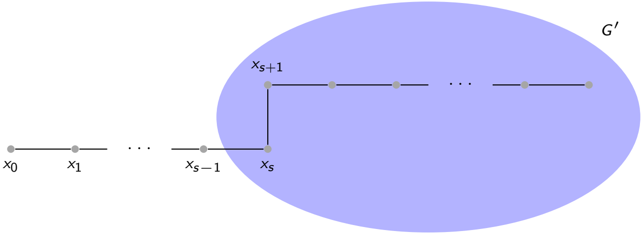
- Given G, construct the height table A. Let $P = x_0x_1$, where x_0x_1 is a max-height edge.
- Extend P to $x_0 \dots x_{s+1}$, where $s = Cn^{1/3}(\lg n)^{2/3}$.

The algorithm



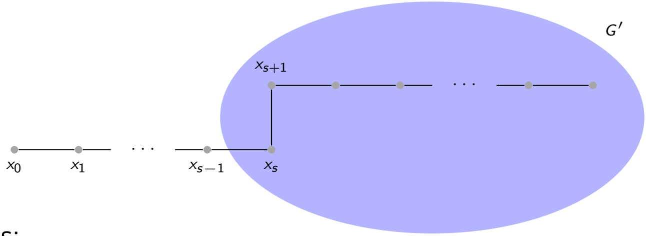
- Given G, construct the height table A. Let $P = x_0 x_1$, where $x_0 x_1$ is a max-height edge.
- Extend P to $x_0 \ldots x_{s+1}$, where $s = Cn^{1/3}(\lg n)^{2/3}$.
- Let $G' = G - \{x_0, \ldots, x_{s-1}\}$.

The algorithm



- Given G, construct the height table A. Let $P = x_0 x_1$, where $x_0 x_1$ is a max-height edge.
- Extend P to $x_0 \dots x_{s+1}$, where $s = Cn^{1/3}(\lg n)^{2/3}$.
- Let $G' = G - \{x_0, \dots, x_{s-1}\}$.
- Recursively find a long mono. path in G' extending $x_s x_{s+1}$.

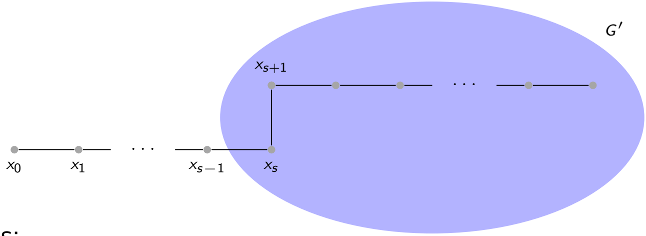
The algorithm



Analysis:

- Extending to $x_0 \ldots x_{s+1}$ uses at most $\binom{s+1}{2}$ rows of A.

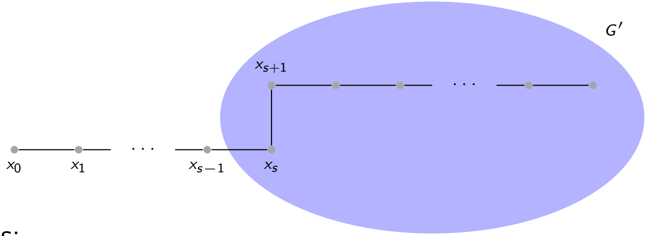
The algorithm



Analysis:

- Extending to $x_0 \ldots x_{s+1}$ uses at most $\binom{s+1}{2}$ rows of A.
- Let $g(n, s)$ be the maximum loss of height of an edge when deleting s vertices from an n-vertex graph.

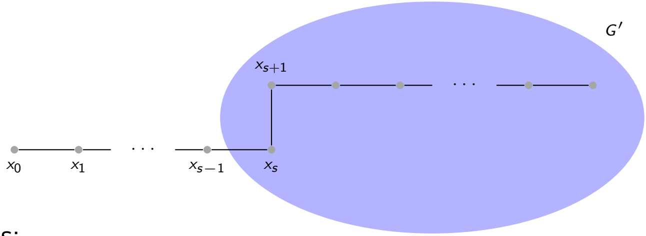
The algorithm



Analysis:

- Extending to $x_0 \dots x_{s+1}$ uses at most $\binom{s+1}{2}$ rows of A.
- Let $g(n, s)$ be the maximum loss of height of an edge when deleting s vertices from an n-vertex graph.
- From G to G', the height of $x_s x_{s+1}$ falls by at most $g(n, s)$.

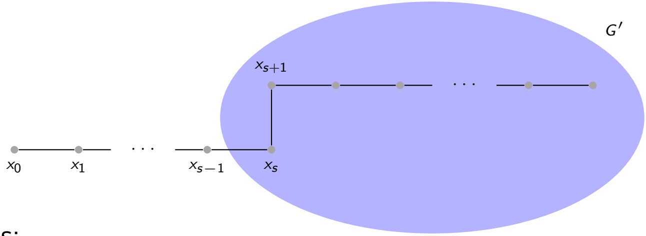
The algorithm



Analysis:

- Extending to $x_0 \dots x_{s+1}$ uses at most $\binom{s+1}{2}$ rows of A.
- Let $g(n,s)$ be the maximum loss of height of an edge when deleting s vertices from an n-vertex graph.
- From G to G', the height of $x_s x_{s+1}$ falls by at most $g(n,s)$.
- Each iteration extends the path by s edges and costs at most $\binom{s+1}{2} + g(n,s)$ in height.

The algorithm



Analysis:

- Extending to $x_0 \dots x_{s+1}$ uses at most $\binom{s+1}{2}$ rows of A.
- Let $g(n,s)$ be the maximum loss of height of an edge when deleting s vertices from an n-vertex graph.
- From G to G', the height of $x_s x_{s+1}$ falls by at most $g(n,s)$.
- Each iteration extends the path by s edges and costs at most $\binom{s+1}{2} + g(n,s)$ in height.

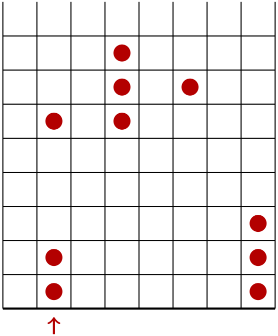
Lemma

If G has average degree d, then $f(G) \geq s \left\lfloor \frac{d/2-1}{\binom{s+1}{2}+g(n,s)} \right\rfloor$.

The (n, s)-token game

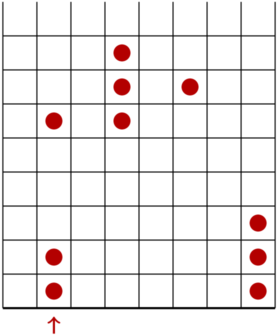
The (n, s)-token game

The (n, s)-token game



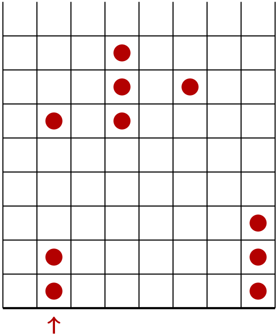
- Some cells contain tokens, others are empty.

The (n, s)-token game



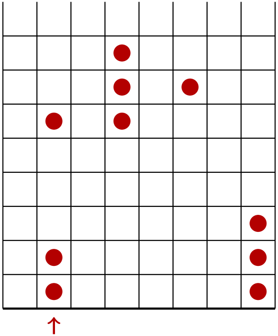
- Some cells contain tokens, others are empty.
- One of the columns is *active*.

The (n, s)-token game



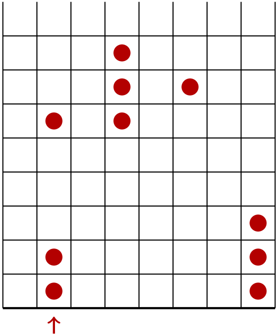
- Some cells contain tokens, others are empty.
- One of the columns is *active*.
- Initially, each column has at most s tokens.

The (n, s)-token game



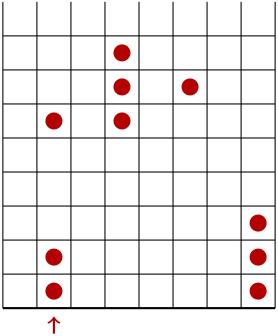
- Some cells contain tokens, others are empty.
- One of the columns is *active*.
- Initially, each column has at most s tokens.
- A token is grounded if all lower cells in the same column contain tokens.

The (n, s)-token game



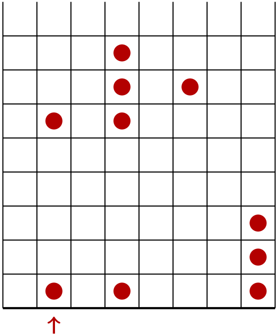
- A step produces a new token array as follows:

The (n, s)-token game



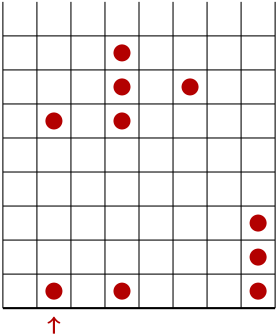
- A step produces a new token array as follows:
  1. The highest grounded token in the active column may move to an empty cell in another column, provided that its height does not increase and no previous step moved a token between these columns.

The (n, s)-token game



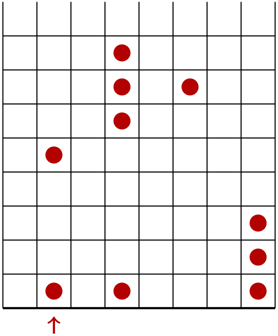
- A step produces a new token array as follows:
    1. The highest grounded token in the active column may move to an empty cell in another column, provided that its height does not increase and no previous step moved a token between these columns.

The (n, s)-token game



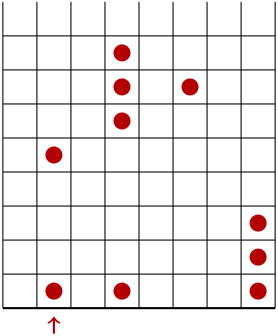
- A step produces a new token array as follows:
    1. The highest grounded token in the active column may move to an empty cell in another column, provided that its height does not increase and no previous step moved a token between these columns.
    2. All ungrounded tokens in the active column drop by one cell.

The (n, s)-token game



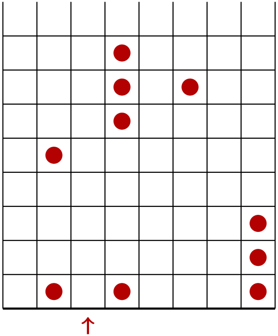
- A step produces a new token array as follows:
  1. The highest grounded token in the active column may move to an empty cell in another column, provided that its height does not increase and no previous step moved a token between these columns.
  2. All ungrounded tokens in the active column drop by one cell.

The (n, s)-token game



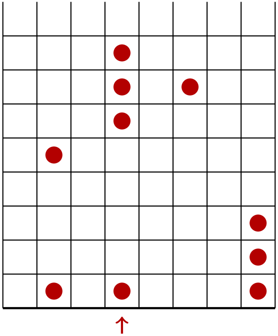
- A step produces a new token array as follows:
  1. The highest grounded token in the active column may move to an empty cell in another column, provided that its height does not increase and no previous step moved a token between these columns.
  2. All ungrounded tokens in the active column drop by one cell.
  3. The active column advances.

The (n, s)-token game



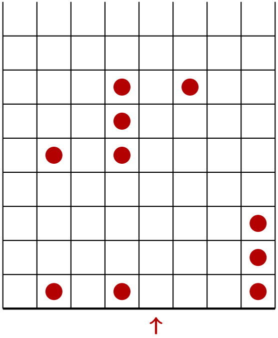
- A step produces a new token array as follows:
  1. The highest grounded token in the active column may move to an empty cell in another column, provided that its height does not increase and no previous step moved a token between these columns.
  2. All ungrounded tokens in the active column drop by one cell.
  3. The active column advances.

The (n, s)-token game



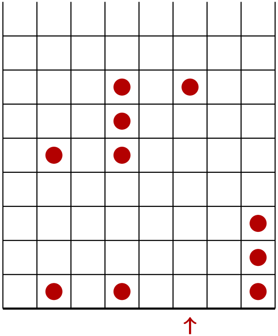
- A step produces a new token array as follows:
  1. The highest grounded token in the active column may move to an empty cell in another column, provided that its height does not increase and no previous step moved a token between these columns.
  2. All ungrounded tokens in the active column drop by one cell.
  3. The active column advances.

The (n, s)-token game



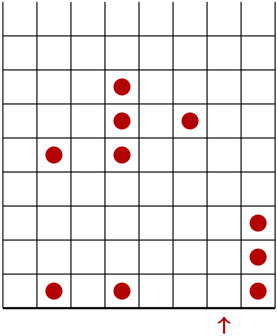
- A step produces a new token array as follows:
  1. The highest grounded token in the active column may move to an empty cell in another column, provided that its height does not increase and no previous step moved a token between these columns.
  2. All ungrounded tokens in the active column drop by one cell.
  3. The active column advances.

The (n, s)-token game



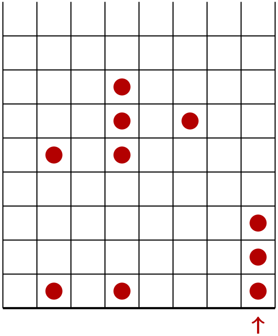
- A step produces a new token array as follows:
  1. The highest grounded token in the active column may move to an empty cell in another column, provided that its height does not increase and no previous step moved a token between these columns.
  2. All ungrounded tokens in the active column drop by one cell.
  3. The active column advances.

The (n, s)-token game



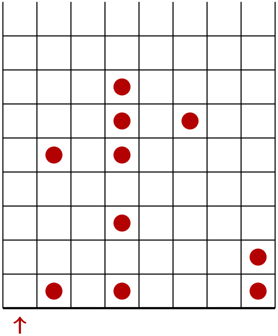
- A step produces a new token array as follows:
  1. The highest grounded token in the active column may move to an empty cell in another column, provided that its height does not increase and no previous step moved a token between these columns.
  2. All ungrounded tokens in the active column drop by one cell.
  3. The active column advances.

The (n, s)-token game



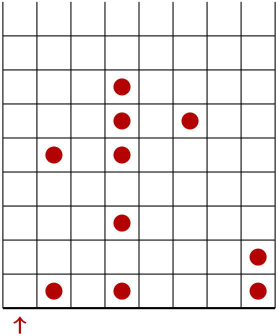
- A step produces a new token array as follows:
  1. The highest grounded token in the active column may move to an empty cell in another column, provided that its height does not increase and no previous step moved a token between these columns.
  2. All ungrounded tokens in the active column drop by one cell.
  3. The active column advances.

The (n, s)-token game



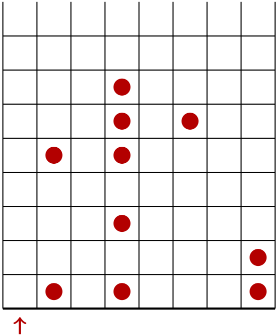
- A step produces a new token array as follows:
  1. The highest grounded token in the active column may move to an empty cell in another column, provided that its height does not increase and no previous step moved a token between these columns.
  2. All ungrounded tokens in the active column drop by one cell.
  3. The active column advances.

The (n, s)-token game



- Let $\hat{g}(n, s)$ be the maximum number of tokens in a column in an (n, s)-token game.

The (n, s)-token game

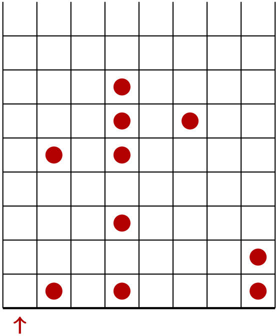


- Let $\hat{g}(n, s)$ be the maximum number of tokens in a column in an (n, s)-token game.

Lemma

$g(n, s) \leq \hat{g}(n - s, s)$

The (n, s)-token game



- Let $\hat{g}(n, s)$ be the maximum number of tokens in a column in an (n, s)-token game.

Lemma

$g(n, s) \leq \hat{g}(n - s, s)$

Lemma

$\Omega(s + \sqrt{ns}) \leq \hat{g}(n, s) \leq O(s + \sqrt{ns} \log n)$

Summary

Lemma

If G has average degree d, then $f(G) \geq s \left\lfloor \frac{d/2-1}{\binom{s+1}{2}+g(n,s)} \right\rfloor$.

Summary

Lemma

If G has average degree d, then $f(G) \geq s \left\lfloor \frac{d/2-1}{\binom{s+1}{2}+g(n,s)} \right\rfloor$.

Lemma

$\Omega(s+\sqrt{ns}) \leq \hat{g}(n,s) \leq O(s+\sqrt{ns}\log n)$

Summary

Lemma

If G has average degree d, then $f(G) \geq s \left\lfloor \frac{d/2-1}{\binom{s+1}{2}+g(n,s)} \right\rfloor$.

Lemma

$\Omega(s+\sqrt{ns}) \leq \hat{g}(n,s) \leq O(s+\sqrt{ns}\log n)$

Theorem

Let G be an n-vertex graph, and let $s = Cn^{1/3}(\lg n)^{2/3}$. If G has average degree d, then

$$f(G) \geq \frac{d}{4s}\left(1-\frac{2}{d}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{4s^2}{d-2}\right).$$

In particular, $f(K_n) \geq (\frac{1}{20}-o(1))(n/\lg n)^{2/3}$.

Summary

Lemma

If G has average degree d, then $f(G) \geq s \left\lfloor \frac{d/2-1}{\binom{s+1}{2}+g(n,s)} \right\rfloor$.

Lemma

$\Omega(s+\sqrt{ns}) \leq \hat{g}(n,s) \leq O(s+\sqrt{ns}\log n)$

Theorem

Let G be an n-vertex graph, and let $s = Cn^{1/3}(\lg n)^{2/3}$. If G has average degree d, then

$$f(G) \geq \frac{d}{4s}\left(1-\frac{2}{d}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{4s^2}{d-2}\right).$$

In particular, $f(K_n) \geq (\frac{1}{20} - o(1))(n/\lg n)^{2/3}$.

Question

Can the bound $g(n,s) \leq O(s+\sqrt{ns}\log n)$ be improved?

Summary

Lemma

If G has average degree d, then $f(G) \geq s \left\lfloor \frac{d/2-1}{\binom{s+1}{2}+g(n,s)} \right\rfloor$.

Lemma

$\Omega(s+\sqrt{ns}) \leq \hat{g}(n,s) \leq O(s+\sqrt{ns}\log n)$

Theorem

Let G be an n-vertex graph, and let $s = Cn^{1/3}(\lg n)^{2/3}$. If G has average degree d, then

$$f(G) \geq \frac{d}{4s}\left(1-\frac{2}{d}\right)\left(1-\frac{1}{2}\right)\left(1-\frac{4s^2}{d-2}\right).$$

In particular, $f(K_n) \geq (\frac{1}{20}-o(1))(n/\lg n)^{2/3}$.

Question

Can the bound $g(n,s) \leq O(s+\sqrt{ns}\log n)$ be improved?

Thank You.