Tight paths in fully directed hypergraphs

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Fully Directed Hypergraphs

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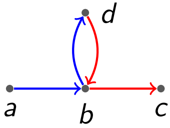
- Example: $V(G) = \{a, b, c, d\}$, $E(G) = \{(a, b, d), (d, b, c)\}$



Paths and Cycles

- The tight path, denoted by $P_n^{(r)}$, is given by:

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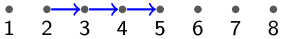

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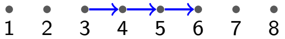

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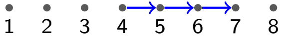

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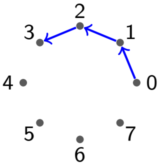

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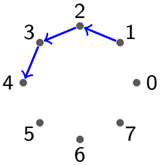

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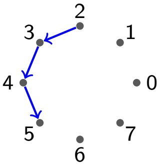

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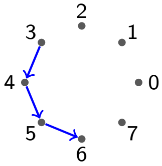

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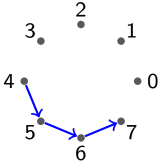

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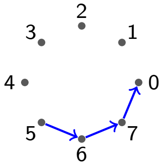

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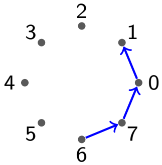

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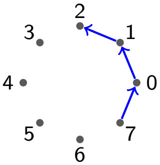

(7, 0, 1, 2)

The extremal function $f(n, r, k)$

- For $0 \leq k \leq r!$, a k-orientation of an r-graph H is a fully directed r-graph G such that for each $e \in E(H)$, exactly k of the $r!$ orderings of the vertices in e are edges in G.

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- Every tournament has a spanning path: $f(n, 2, 1) = n$

Warmup: $f(n, r, r! - 1) = n$

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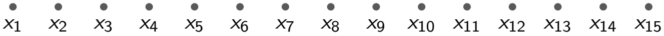

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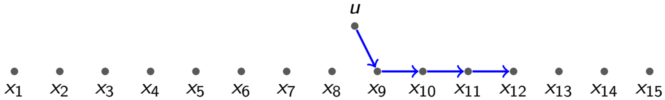

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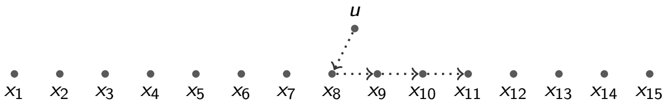

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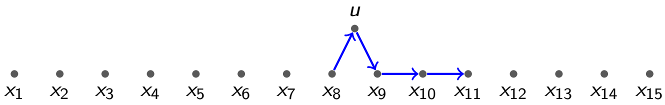

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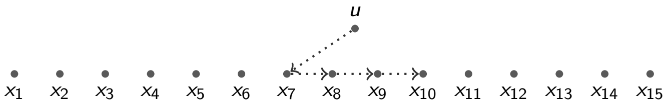

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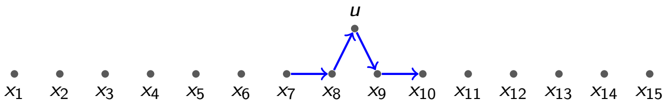

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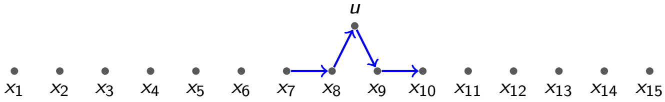

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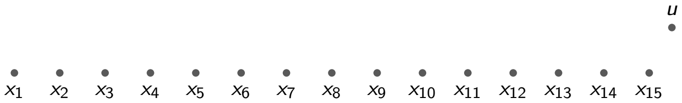

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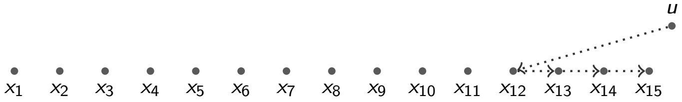

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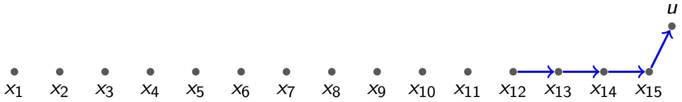
- Let $u \in V(G)$ and apply induction to obtain a spanning path $x_1 \cdots x_{n-1}$ in $G - u$.



- Suppose $(u, x_i, \ldots, x_{i+r-2}) \in E(G)$ for some i.
- Let i be the least such integer.
- Note $(u, x_{i-1}, \ldots, x_{i+r-3}) \notin E(G)$.
- So $(x_{i-1}, u, x_i, \ldots, x_{i+r-3}) \in E(G)$.
- We may insert u before x_i.
- If no such i exists, then append u at the end.

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  - equals n?
- All but the first are open for general r.

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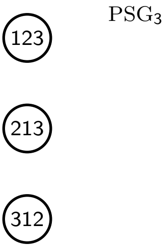

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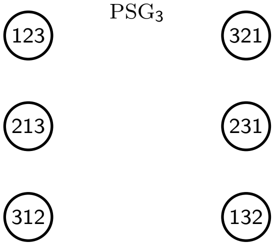

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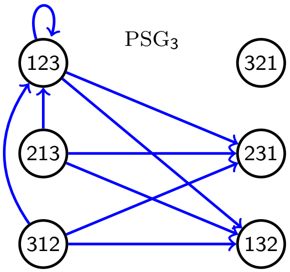

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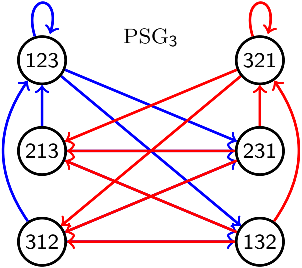

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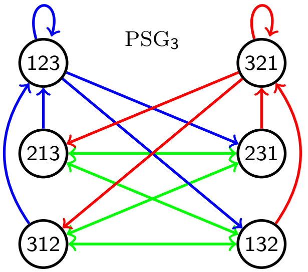

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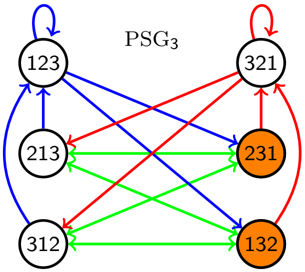

- Note: $\{132, 231\}$ is a max. acyclic set in PSG_3.

Growing paths and spanning paths

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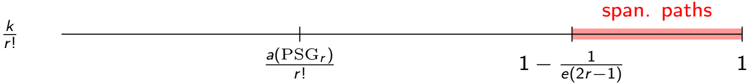

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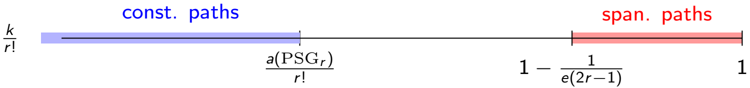

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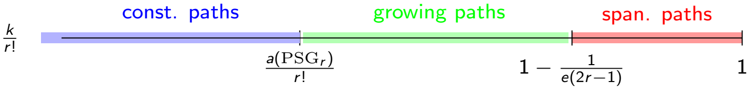

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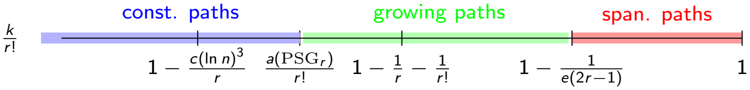

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The case $r = 3$

- Recall: $a(\mathrm{PSG}_3) = 2$.
- So $f(n,3,2) = O(1)$ and $f(n,3,3) = \omega(1)$.
- In fact, we get $f(n,3,2) \leq 3$ and $f(n,3,3) \geq \Omega(\log\log n)$.
- Warmup: $f(n,3,5) = f(n,3,6) = n$.
- Interesting cases: $k = 3$ and $k = 4$.

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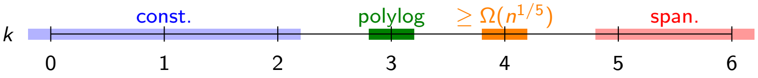

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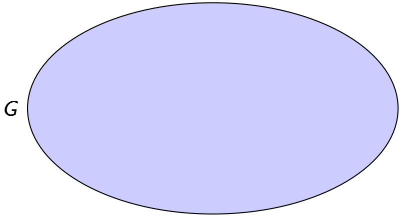

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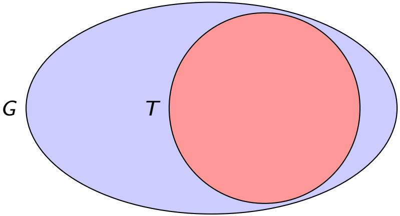

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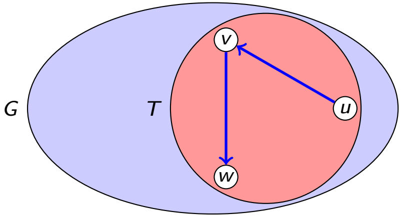

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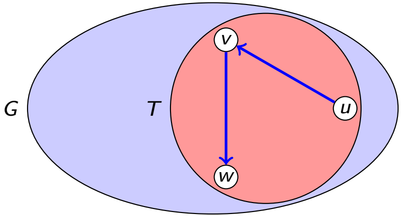

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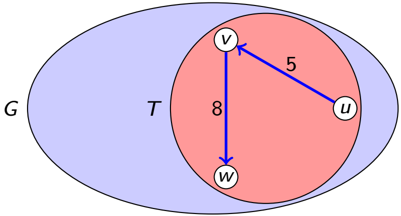

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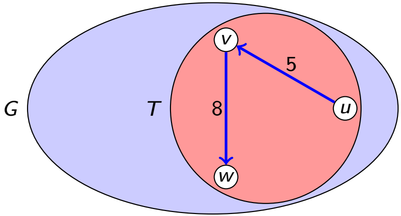

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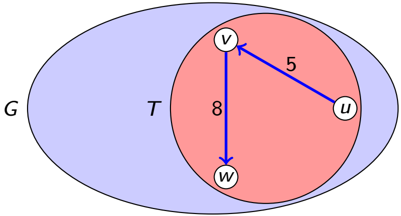

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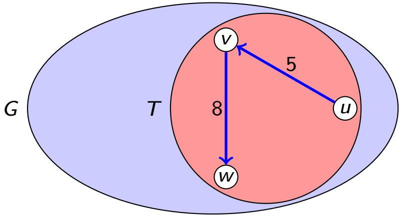

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- What is the threshold on k for (r, k)-tournaments to have polynomial paths? spanning paths?

Thank You.