Tight paths in fully directed hypergraphs

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Abstract

It is well-known that every tournament has a spanning path. We consider hypergraph analogues. In an *r-uniform fully directed hypergraph*, or *r-digraph*, every edge is a list or r distinct vertices. An (r,k)-tournament is an r-digraph G such that for every r-set S of vertices in G, exactly k of the orderings of S are edges in G. A *directed tight path* is an r-digraph G whose vertices can be ordered so that the intervals of size r are the edges in G. Let $f(n,r,k)$ be the maximum s such that every n-vertex (r,k)-tournament contains a tight path on s vertices. Since every tournament has a spanning path, we have $f(n,2,1)=n$.

In this paper, we show that the minimum k such that $f(n,r,k)$ tends to infinity with n is in the interval $\left[\left(1-\frac{1}{r}-O(\frac{\log r}{r^2\log\log r})\right)r!,\ \left(1-\frac{1}{r}-\frac{\varphi(r)-1}{r!}\right)r!\right]$ where $\varphi(r)$ is the Euler Totient Function, and we find the exact value when $r\le 5$. We also show that $\Omega(\sqrt{\log n/\log\log n})\le f(n,3,3)\le O(\log n)$ and $f(n,3,4)\ge\Omega(n^{1/5})$.

1 Introduction

In a *directed graph* or *digraph*, each edge is an ordered pair of vertices. There are several natural generalizations to hypergraphs, such as partitioning each edge into a set of head vertices and a set of tail vertices. In this paper, an edge in a *fully directed hypergraph* is an (ordered) list of distinct vertices. To our knowledge, only a few results have been obtained in this model [2, 4].

An *r-graph* is an r-uniform hypergraph, and an *r-digraph* is an r-uniform fully directed hypergraph. Many concepts from directed graphs extend naturally to the fully directed hypergraph setting. If F and G are r-digraphs, we say that F is a *subgraph* of G, denoted $F\subseteq G$ if there is an injection $h\colon V(F)\to V(G)$ such that $(u_1,\dots,u_r)\in E(F)$ implies $(h(u_1),\dots,h(u_r))\in E(G)$.

For each r-digraph F, we define the *Turán number*, denoted $\mathrm{ex}(n,F)$, to be the maximum number of edges in an n-vertex r-digraph G that does not contain F. Similarly, if $\mathcal{F}$ is a family of r-digraphs, then we define $\mathrm{ex}(n,\mathcal{F})$ to be the maximum number of edges in an n-vertex r-digraph G such that $F\not\subseteq G$ for each $F\in\mathcal{F}$. Brown and Simonovits [2] studied a generalization of this model in 1984 which allows edges to have multiplicity greater than 1. Among other results, they proved that the Turán number for an infinite family of graphs is approximated by a finite subfamily. That is, if $\mathcal{F}$ is a family of r-digraphs with uniformly bounded edge multiplicities, then for each positive ε there exists a finite $\mathcal{F}_0\subseteq\mathcal{F}$ such that $\mathrm{ex}(n,\mathcal{F}_0)-\varepsilon n^r\le\mathrm{ex}(n,\mathcal{F})\le\mathrm{ex}(n,\mathcal{F}_0)$.

For $k\le r!$, a *k-orientation* of an r-graph G is an r-digraph G' such that $V(G')=V(G)$ and for each $e\in E(G)$ exactly k of the orderings of vertices in e are edges in G'. An *(r,k)-tournament* is a k-orientation of a complete r-graph. In our language, the usual notion of a tournament is a $(2,1)$-tournament. With r fixed and k increasing from 0 to $r!$, an (r,k)-tournament evolves from the empty r-digraph to the complete r-digraph. For a fixed r, it is natural to ask for the minimum value of k such that every (r,k)-tournament has some structure or property.

In a list of objects $(a_1,\dots,a_t)$, an *interval* is a sublist of consecutive entries. An *r-interval* is an interval of size r. A *directed tight path* is an r-digraph whose vertices can be ordered so that the r-intervals are the edges in G. The s-vertex directed tight path is denoted $P_s^{(r)}$ and we write $P_s^{(r)}=v_1\cdots v_s$ when $v_1,\dots,v_s$ is the ordering of $V(P_s^{(r)})$ whose r-intervals are the edges in $P_s^{(r)}$. Similarly, a *directed tight cycle* is an r-digraph G whose vertices can be arranged cyclically so that the intervals of size r are the edges in G. For $s\ge r$, the s-vertex directed tight cycle is denoted $C_s^{(r)}$ and we write $C_s^{(r)}=\langle v_1,\dots,v_s\rangle$ when $v_1,\dots,v_s$ is a cyclic arrangement of $V(C_s^{(r)})$ whose r-intervals are the edges in $C_s^{(r)}$. In this paper, a *path* and a *cycle* refer to tight paths and tight cycles unless otherwise stated.

Let $f(n,r,k)$ be the maximum integer s such that every n-vertex (r,k)-tournament contains a copy of $P_s^{(r)}$ as a subgraph. It is well-known that each $(2,1)$-tournament contains a spanning path, and so $f(n,2,1)=n$. Note that for $n \geq r-1$, we have $r-1 = f(n,r,0) \leq f(n,r,1) \leq \cdots \leq f(n,r,r!) = n$. We are interested in several natural thresholds. In terms of r, how large does k need to be before $f(n,r,k)$ tends to infinity with n, is linear in n, or equals n? These are the *growing*, *linear*, and *spanning path thresholds*, respectively.

In Section 2, we obtain bounds on the thresholds for k for various growth behaviors of $f(n,r,k)$. Our main result is to obtain the exact threshold on k for $f(n,r,k)=\omega(1)$ in terms of a simpler combinatorial problem on the *pattern shift graph*, and we use this to show that this threshold is between $\left(1-\frac{1}{r}-O(\frac{\log r}{r^2 \log\log r})\right)r!$ and $(1-\frac{1}{r}-\frac{\varphi(r)-1}{r!})r!$, where $\varphi(r)$ is the Euler Totient Function $\varphi(r) = |\{d \in [r]\colon \gcd(d,r)=1\}|$ (see Corollary 13). Also, if $k > (1-1/r)r!$, then every (r,k)-tournament has paths of linear size (see Theorem 18), and if $k > (1-\frac{1}{4(r-1)})r!$, then every (r,k)-tournament has spanning paths (see Theorem 21).

In Section 3, we consider fixed small r. We have that $f(n,3,k)=O(1)$ when $k \leq 2$ and $f(n,3,k)=n$ when $k \geq 5$. Finding sharp bounds on $f(n,3,k)$ is interesting when $k \in \{3,4\}$. We show that $(\log n)^{1/4-o(1)} \leq f(n,3,3) \leq O(\log n)$ and that $f(n,3,4) \geq \Omega(n^{1/5})$.

In Section 4, we consider density thresholds for the emergence of tight paths in r-digraphs. The *density* of an n-vertex r-digraph G equals $|E(G)|/n_{(r)}$, where $n_{(r)}$ is the *falling factorial* given by $n_{(r)} = n(n-1)\cdots(n-r+1)$. The *density threshold* for growing tight paths d^* is the infimum (in fact, minimum when $r \geq 2$) of the set of $d \in [0,1]$ such that every sufficiently large n-vertex r-digraph with density at least d has a copy of $P_s^{(r)}$, where s grows with n. In Lemma 31, we show that $d^* \geq 1-\frac{1}{r}$, and in Corollary 33, we show that $d^* \leq 1-\frac{1}{r}$.

The growing path threshold for (r,k)-tournaments is one more than the maximum size of an acyclic set of vertices in a directed graph that we call the *pattern-shift graph* (see Theorem 2). The *t-prefix* of a list is the sublist of its first t elements. Similarly, the *t-suffix* is the sublist of its last t elements. Let $a=(a_1,\ldots,a_s)$ and $b=(b_1,\ldots,b_s)$, and suppose the entries within each list are distinct. We say that a and b are *order isomorphic* or *pattern-match* if, for $i<j$, we have $a_i<a_j$ if and only if $b_i<b_j$. The *pattern-shift graph* of order r, denoted PSG_r, is the digraph whose vertices are the permutations of $[r]$ with an edge from a to b if and only if the $(r-1)$-suffix of a and the $(r-1)$-prefix of b pattern-match. See Figure 1 for PSG_3. The pattern-shift graph is an analogue of the well-known de Bruijn graph. Close variants of the pattern-shift graph were previously investigated by Asplund and Fox [1] and by Chung, Diaconis, and Graham [5].

2 Thresholds for Tournaments

2.1 Constant Path Thresholds

Let G be an (r,k)-tournament. When $k=0$, the r-digraph G has no edges and hence has paths of size at most $r-1$ and length 0. When $k>0$, edges are present and so G has paths of size at least r and length at least 1. Our next proposition shows that longer paths are not forced even when k is as large as $r!/3$.

Proposition 1. *If $0<k\leq \frac{1}{3}r!$, then $f(n,r,k)=r$.*

Proof. Taking a single edge in an (r,k)-tournament shows that $f(n,r,k)\geq r$. For the upper bound, note that $1\leq k\leq \frac{1}{3}r!$ implies that $r\geq 3$ and we may assume that $k=r!/3$. We give an (r,k)-tournament avoiding $P_{r+1}^{(r)}$. We use $[n]$ for our vertex set and we let $(u_1,\cdots,u_r)$ be an edge if and only if $u_2=\max\{u_1,u_2,u_3\}$. Note that $v_1\cdots v_{r+1}$ does not form a copy of $P_{r+1}^{(r)}$ since the first r-tuple requires $v_2>v_3$ and the second r-tuple requires $v_2<v_3$. $\square$

Let t be a small, fixed non-negative integer and let r be a positive integer. The *threshold for paths of length t* is the minimum k such that for sufficiently large n, we have that $f(n,r,k)\geq r-1+t$. Note that when k reaches the threshold value for paths of length t, every sufficiently large (r,k)-tournament contains a path of length t, but when k is less than this threshold value there are arbitrarily large (r,k)-tournaments that avoid paths of length t. The threshold for paths of length 0 occurs at $k=0$, and the threshold for paths of length 1 occurs at $k=1$. Proposition 1 shows that the threshold for paths of length 2 is more than $r!/3$.

Our next aim is to find the threshold for paths of length t exactly in terms of an optimization problem in a particular directed graph.

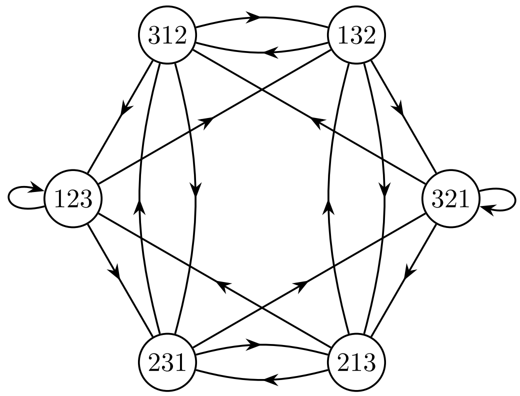

Figure 1: PSG_3

Note that the PSG_r is regular with in-degree and out-degree both r, and the loops in PSG_r are at the identity permutation and its reverse. Note that for each list $(a_1, \ldots, a_r)$ with distinct integer entries, there is exactly one permutation of $[r]$ that matches patterns with $(a_1, \ldots, a_r)$. We call this permutation of $[r]$ the *canonical pattern* of $(a_1, \ldots, a_r)$ and denote it by $\rho(a_1, \ldots, a_r)$.

Let $t \geq 1$ and let G be a digraph. Let $a_t(G)$ be the maximum size of a set of vertices in G inducing a subgraph with no walk of size t. Note that when $|V(G)| = n$, we have $0 = a_1(G) \leq a_2(G) \leq \ldots \leq a_{n+1}(G) = a(G)$, where $a(G)$ is the maximum size of a set of vertices in G inducing an acyclic subgraph. Note that $a_2(G) = \alpha(G)$, where $\alpha(G)$ is the maximum size of a set of vertices containing no edges or loops. Note that when $G = \mathrm{PSG}_3$, we have $a_1(G) = 0$ and $a_t(G) = 2$ for $t \geq 2$ (achieved by, for example, $\{132, 231\}$).

Theorem 2. *Let $r \geq 1$ and $t \geq 1$. The threshold for paths of length t is $1 + a_t(\mathrm{PSG}_r)$. In other words, for n large enough, we have $f(n, r, k) < r - 1 + t$ if $k = a_t(\mathrm{PSG}_r)$ and $f(n, r, k) \geq r - 1 + t$ if $k = 1 + a_t(\mathrm{PSG}_r)$.*

Proof. First, we show that $f(n, r, k) < r - 1 + t$ when $k = a_t(\mathrm{PSG}_r)$. Let S be a set of $a_t(\mathrm{PSG}_r)$ vertices in PSG_r that induces a subgraph containing no walk of size t. For large n, we construct an (r, k)-tournament G on $[n]$ by putting $(u_1, \ldots, u_r) \in E(G)$ if and only if $\rho(u_1, \ldots, u_r) \in S$. If $v_1 \ldots v_m$ is a walk in G, then each r-interval $(v_{j+1}, \ldots, v_{j+r})$ satisfies $\rho(v_{j+1}, \ldots, v_{j+r}) \in S$. Moreover, it is clear that the last $r-1$ entries of $\rho(v_j, \ldots, v_{j+r-1})$ and the first $r-1$ entries of $\rho(v_{j+1}, \ldots, v_{j+r})$ both pattern-match $(v_{j+1}, \ldots, v_{j+r-1})$.

Let $w_j = \rho(v_j, \ldots, v_{j+r-1})$ and let $m' = m - r + 1$. Note that $w_1, \ldots, w_{m'}$ is a walk in the subgraph of PSG_r induced by S of size m'. It follows that $m' < t$ and so $m < r - 1 + t$, implying the desired upper bound on $f(n, r, k)$.

Next, we show that for sufficiently large n, we have $f(n, r, k) \geq r - 1 + t$ when $k = 1 + a_t(\mathrm{PSG}_r)$. Let G be an (r, k)-tournament on $[n]$. For each r-set A, let $f(A) = \{\rho(e) \colon e \in E(G) \text{ and } e \text{ is an ordering of } A\}$ and note that since G is an (r, k)-tournament, for each r-set A, we have that $f(A)$ is a set of k vertices in PSG_r. Hence $f \colon \binom{[n]}{r} \to \binom{V(\mathrm{PSG}_r)}{k}$ is a $\binom{r!}{k}$-edge-coloring of the complete r-graph on $[n]$. Let $m = r - 1 + t$. By Ramsey's Theorem [7], since n is sufficiently large, we obtain an m-vertex subgraph G_0 which is monochromatic in some color $S \in \binom{V(\mathrm{PSG}_r)}{k}$. Since $|S| = k > a_t(\mathrm{PSG}_r)$, it follows that there is a walk $w_1 \ldots w_t$ in the subgraph of PSG_r induced by S. By induction on t, we obtain a list of distinct real numbers $x_1, \ldots, x_m$ with $m = r - 1 + t$ such that $\rho(x_j, \ldots, x_{j+r-1}) = w_j$ for $1 \leq j \leq t$. For $t = 1$, we may simply take $(x_1, \ldots, x_r) = w_1$. For $t \geq 2$, we obtain $x_1, \ldots, x_{m-1}$ inductively and then select x_m distinct from $x_1, \ldots, x_{m-1}$ so that $\rho(x_t, \ldots, x_m) = w_t$. This is possible since $\rho(x_{t-1}, \ldots, x_{m-1}) = w_{t-1}$ and $w_{t-1}w_t \in E(\mathrm{PSG}_r)$, implying that $(x_t, \ldots, x_{m-1})$ pattern-matches the first $r - 1$ entries of w_t.

We claim that for the ordering $v_1 \ldots v_m$ of $V(G_0)$ which pattern-matches $x_1, \ldots, x_m$, we have that $v_1 \ldots v_m$ is a path. Indeed, for $1 \leq j \leq t$ we have that $(v_j, \ldots, v_{j+r-1})$ pattern-matches $(x_j, \ldots, x_{j+r-1})$, and $\rho(x_j, \ldots, x_{j+r-1}) = w_j \in S$. Since $f(A) = S$ for each r-set of vertices in G_0, including $A = \{v_j, \ldots, v_{j+r-1}\}$, it follows that $(v_j, \ldots, v_{j+r-1}) \in E(G_0)$. Hence G contains a path of size m and so $f(n, r, k) \geq m = r - 1 + t$ when n is sufficiently large. $\square$

In general, computing $a_t(\mathrm{PSG}_r)$ seems challenging. It is convenient to define the complementary *transversal number*, denoted $\tau_t(G)$, to be the minimum size of a set of vertices in G which intersects every walk in G of size t. Since the complement of a set of vertices avoiding all walks of size t intersects every such walk, always $a_t(G) + \tau_t(G) = |V(G)|$. Similarly, let $\tau(G)$ be the minimum size of a set of vertices in G which intersects every cycle in G, and note that $\tau(G) + a(G) = |V(G)|$.

Our next aim is to find $a_t(\mathrm{PSG}_r)$ exactly when t divides $r-1$.

Proposition 3. $\tau_t(\mathrm{PSG}_r) \le \left(\frac{1}{r} + \left\lceil \frac{r-1}{t} \right\rceil \cdot \frac{1}{r}\right) r!$

Proof. Let $S \subseteq V(\mathrm{PSG}_r)$ be the set of all vertices $(x_0, \ldots, x_{r-1})$ with the property that the index i satisfying $x_i = 1$ satisfies either $i = r-1$ or $i \equiv 0 \pmod t$. Note that besides the last index $r-1$, there are $\left\lceil \frac{r-1}{t} \right\rceil$ indices i with $0 \le i \le r-2$ such that $i \equiv 0 \pmod t$. Including the last index, S collects all permutations of $[r]$ which have the element 1 in one of $1 + \left\lceil \frac{r-1}{t} \right\rceil$ positions. It follows that $|S| = \left(1 + \left\lceil \frac{r-1}{t} \right\rceil\right)(r-1)!$.

Suppose that $uv \in E(\mathrm{PSG}_r)$ with $u = (x_0, \ldots, x_{r-1})$ and $v = (y_0, \ldots, y_{r-1})$. Let i and j be the indices satisfying $u_i = 1$ and $v_j = 1$. We claim that either $i = 0$, $j = r-1$, or $j = i-1$. Indeed, if $i > 0$ and $j < r-1$, then the minimum element in u is in the last $r-1$ positions and the minimum element in v is in the first $r-1$ positions. Since $(u_1, \ldots, u_{r-1})$ and $(v_0, \ldots, v_{r-2})$ pattern-match, it follows that $j = i-1$.

Let $u_1 \ldots u_k$ be a walk in PSG_r which is disjoint from S, and let z_i index the position of 1 in u_i. Since $u_1 \ldots u_k$ is disjoint from S, we have that $0 < z_i < r-1$, and it follows that z_i is strictly decreasing. Since no z_i is congruent to 0 modulo t, it follows that $k \le t-1$. Hence S intersects every walk in PSG_r of size t. □

Proposition 4. *If $t \le r$, then $\tau_t(\mathrm{PSG}_r) \ge \frac{1}{r}\left\lceil \frac{r}{t} \right\rceil \cdot r!$.*

Proof. Note that the cyclic shift operation partitions PSG_r into a family $\mathcal{C}$ of $(r-1)!$ cycles of size r. Let S be a set of vertices that intersects each walk in PSG_r of size t, and let $C \in \mathcal{C}$. If $k = |S \cap V(C)|$, then we have $r - k \le k(t-1)$ since removing the k vertices in $S \cap V(C)$ from C leaves at most k path components, each with at most $t-1$ vertices. It follows that each $C \in \mathcal{C}$ contains at least $\lceil r/t \rceil$ vertices in S, and since $\mathcal{C}$ is a disjoint family with $(r-1)!$ cycles, we have $|S| \ge \left\lceil \frac{r}{t} \right\rceil (r-1)!$. □

Theorem 5. *If t divides $r-1$, then $\tau_t(\mathrm{PSG}_r) = \left(\frac{1}{r} + \frac{1}{t} - \frac{1}{tr}\right) r!$ and $a_t(\mathrm{PSG}_r) = \left(1 - \frac{1}{r} - \frac{1}{t} + \frac{1}{tr}\right) r!$.*

Proof. Suppose that $r-1 = tk$ for some integer k. By Proposition 3, we have $\tau_t(\mathrm{PSG}_r) \le \left(\frac{1}{r} + \left\lceil \frac{r-1}{t} \right\rceil \cdot \frac{1}{r}\right) r! = \frac{1+k}{r} r!$. By Proposition 4, we have $\tau_t(\mathrm{PSG}_r) \ge \frac{1}{r}\left\lceil \frac{r}{t} \right\rceil \cdot r! = \frac{1+k}{r} r!$. It follows that $\tau_t(\mathrm{PSG}_r) = \frac{1+k}{r} r! = \left(\frac{1}{r} + \frac{1}{t} - \frac{1}{tr}\right) r!$. □

2.2 Growing Path Threshold

Let r be a positive integer. The *threshold for growing paths* is the minimum k such that we have that $f(n, r, k) \to \infty$ as $n \to \infty$.

Theorem 6. *Let r be a positive integer. The threshold for growing paths is $1 + a(\mathrm{PSG}_r)$. That is, we have*

$$f(n, r, k) = \begin{cases} O(1) & \text{if } k \le a(\mathrm{PSG}_r) \\ \omega(1) & \text{if } k > a(\mathrm{PSG}_r). \end{cases}$$

Proof. Let $k = a(\mathrm{PSG}_r)$. We set $t = r! + 1$, so that $k = a(\mathrm{PSG}_r) = a_t(\mathrm{PSG}_r)$. By Theorem 2, we have $f(n, r, k) < r - 1 + t$, implying that $f(n, r, k) = O(1)$. Now, let $k = 1 + a(\mathrm{PSG}_r)$. By Theorem 2, for each t, there exists n such that $f(n, r, k) \ge r - 1 + t$, implying that $f(n, r, k) \to \infty$ as $n \to \infty$. □

Our next proposition is simple but very useful.

Proposition 7. *Let uv be an edge in PSG_r with $u = (u_1, \ldots, u_r)$ and $v = (v_1, \ldots, v_r)$. Let i be the index such that $u_i = r$ and let j be the index such that $v_j = r$. Either $j = i-1$, $j = r$, or $i = 1$. Consequently every cycle in PSG_r contains a vertex $(u_1, \ldots, u_r)$ such that $u_1 = r$ or $u_r = r$.*

Proof. Suppose that $i > 1$ and $j < r$. Since $r = u_i = \max(u_2, \ldots, u_r)$ and $uv \in E(\mathrm{PSG}_r)$, it follows that $v_{i-1} = \max(v_1, \ldots, v_{r-1})$. Since $v_r \ne r$, we conclude that $v_{i-1} = r$ and so $j = i-1$. Since the index of the position containing r cannot decrease indefinitely along edges in a cycle, the rest of the proposition follows. □

Theorem 8. *Let $r \ge \max\{3t+2, 3\}$. We have $\tau(\mathrm{PSG}_r) \le r!\left(\frac{1}{r} + \frac{1}{rt!} + \frac{6(3t+2)}{r(r-1)}\right)$. Consequently, $\tau(\mathrm{PSG}_r) \le r!\left(\frac{1}{r} + O\left(\frac{\log r}{r^2 \log\log r}\right)\right)$.*

Proof. We construct a cycle transversal as follows. Let $A = \{1,\dots,t+1\} \cup \{r-2t,\dots,r\}$, so that A contains the indices of the first $t+1$ positions and the last $2t+1$ positions in permutations of $[r]$. Let S_0 be the set of permutations $(u_1,\dots,u_r)$ of $[r]$ in which $(u_i,u_j) = (r-1,r)$ for some i,j with $\{i,j\} \subseteq A$ and $\{i,j\} \cap \{1, r-t, r\} \neq \varnothing$. We use the values r and $r-1$ as bookmarks for a pattern, and we show that this pattern persists along cycles in $\mathrm{PSG}_r - S_0$.

Fix a linear ordering $\preceq$ on the permutations of $[t]$. Given a vertex $u \in V(\mathrm{PSG}_r)$ such that $u = (u_1,\dots,u_r)$ and $u_j = \ell$ for some $j \le r-t$, the *ℓ-cluster* of u, denoted $\lambda_\ell(u)$, is the canonical pattern $\rho(u_{j+1},\dots,u_{j+t})$. Let S_1 be the set of all $u \in V(\mathrm{PSG}_r) - S_0$ such that $u = (u_1,\dots,u_r)$, $u_{r-t} = r$, and $\lambda_{r-1}(u) \succeq \lambda_r(u)$. Similarly, let S_2 be the set of all $u \in V(\mathrm{PSG}_r) - S_0$ such that $u = (u_1,\dots,u_r)$ with $u_1 = r$ and $\lambda_r(u) \succeq \lambda_{r-1}(u)$.

Let $S = S_0 \cup S_1 \cup S_2$. We claim that S is a cycle transversal. Let B be the set of $(u_1,\dots,u_r) \in V(\mathrm{PSG}_r)$ such that $(u_i,u_j) = (r-1,r)$ for some i,j with $\{i,j\} \cap \{1,\dots,r-t\} \neq \varnothing$. Note that B is the set of vertices u in PSG_r such that either $\lambda_r(u)$ or $\lambda_{r-1}(u)$ is defined. We define a function g from B to the permutations on $[t]$ as follows. Given $u \in B$ with $u = (u_1,\dots,u_r)$, if the index j with $u_j = r$ satisfies $j \le r-t$, then we set $g(u) = \lambda_r(u)$. Otherwise, we set $g(u) = \lambda_{r-1}(u)$.

Let C be a cycle in PSG_r. Next, we show that either C contains a vertex in $S_0 \cup B$. By Proposition 7, there exists $u \in V(C)$ with $u = (u_1,\dots,u_r)$ such that $u_1 = r$ or $u_r = r$. If $u_1 = r$, then $u \in B$. Otherwise, $u_r = r$. Let i be the index with $u_i = r-1$. If $i \ge r-2t$, then $u \in S_0$. Otherwise $u \in B$. Hence every cycle in $\mathrm{PSG}_r - S_0$ contains a vertex in B.

Let $u = (u_1,\dots,u_r)$ and $v = (v_1,\dots,v_r)$ with $(u_i,u_j) = (r-1,r)$. We show that if $uv \in E(\mathrm{PSG}_r)$ with $u,v \notin S$ and $u \in B$, then $v \in B$ and $g(u) \preceq g(v)$. Moreover, when $j=1$ or $j = r-t+1$, we have $g(u) \prec g(v)$.

Case 1: $j \le r-t$. Note that $g(u) = \lambda_r(u)$.

Subcase 1(a): Suppose that $j > 1$. Since u_j is the maximum entry in $(u_2,\dots,u_r)$ and $uv \in E(\mathrm{PSG}_r)$, it follows that v_{j-1} is the maximum entry in $(v_1,\dots,v_{r-1})$. If also $v_r < r$, then we have that $v_{j-1} = r$, and so $g(v) = \lambda_r(v) = \lambda_r(u) = g(u)$. Otherwise, $j > 1$ and $v_r = r$, and since v_{j-1} is the maximum entry in $(v_1,\dots,v_{r-1})$, we have that $v_{j-1} = r-1$. We have $g(v) = \lambda_{r-1}(v) = \lambda_r(u) = g(u)$.

Subcase 1(b): Suppose $j = 1$. We show that $g(u) \prec g(v)$. Note that since $j=1$ and $u \notin S_0$, we have that $i \notin A$, implying $t+1 < i < r-2t$. Since u_i is the maximum entry in $(u_2,\dots,u_r)$, it follows that v_{i-1} is the maximum entry in $(v_1,\dots,v_{r-1})$. If $v_r < r$, then we have $v_{i-1} = r$ and so $g(v) = \lambda_r(v) = \lambda_{r-1}(u)$. Since $u_1 = r$ and $u \notin S_2$, we have $\lambda_{r-1}(u) \succ \lambda_r(u) = g(u)$, and so $g(v) \succ g(u)$. If both $u_1 = r$ and $v_r = r$, then v is obtained from u by a cyclic shift, and we have $g(v) = \lambda_{r-1}(v) = \lambda_{r-1}(u) \succ \lambda_r(u) = g(u)$.

Case 2: $j > r-t$. Note that since $u \in B$ but $\lambda_r(u)$ is undefined, we have that $i \le r-t$ and $g(u) = \lambda_{r-1}(u)$. Also, $v_r \le r-2$, since $v_r \in \{r-1,r\}$ and $v_{j-1} \ge u_j - 1 \ge r-1$ would contradict $v \notin S_0$. Since $v_r < r$, it follows that $v_{j-1} = r$. Since $u \notin S_0$ and r appears in the last t positions of u, it follows that $i \ge 2$. Also, since u_i is the second largest element in $(u_2,\dots,u_r)$, it follows that v_{i-1} is the second largest element in $(v_1,\dots,v_{r-1})$. Together with $v_r \le r-2$, it follows that $v_{i-1} = r-1$. So $(v_{i-1},v_{j-1}) = (r-1,r)$ and therefore $\lambda_{r-1}(u) = \lambda_{r-1}(v)$.

Subcase 2(a): If $j-1 > r-t$, then since $v_{j-1} = r$ we have that $g(v) = \lambda_{r-1}(v) = \lambda_{r-1}(u) = g(u)$.

Subcase 2(b): Otherwise $j-1 = r-t$. We show that $g(u) \prec g(v)$. Since the index of the position in v with value r is $j-1$ and $j-1 = r-t$, it follows that $g(v) = \lambda_r(v)$. Also, since $v \notin S_1$ and $v_{r-t} = r$, we have that $\lambda_{r-1}(v) \prec \lambda_r(v)$. It follows that $g(v) = \lambda_r(v) \succ \lambda_{r-1}(v) = \lambda_{r-1}(u) = g(u)$.

Suppose for a contradiction that C is a cycle in $\mathrm{PSG}_r - S$. Since C contains a vertex $u \in B$, it follows that $V(C) \subseteq B$. Also, since $g(x) \preceq g(y)$ for each $xy \in E(C)$, it follows that $g(x) = g(y)$ for all $x,y \in V(C)$. Note that if $uv \in E(C)$ with $u = (u_1,\dots,u_r)$, $v = (v_1,\dots,v_r)$, and $u_j = r$, then either $j = 1$, or $v_{j-1} = r$, or $v_r = r$. The case $j=1$ is excluded, since then $g(u) \prec g(v)$. It follows that C has a vertex with r in the last position. Choose $w \in V(C)$ with $w = (w_1,\dots,w_r)$ such that the index of the position j with $w_j = r$ is minimized subject to $j \ge r-t+1$. Let w' be the successor of w in C, with $w' = (w'_1,\dots,w'_r)$. Since $j = r-t+1$ would imply $g(w) \prec g(w')$, we have $j > r-t+1$. Note that $w'_r < r$, or else $w'_r = r$ and $w'_{j-1} = r-1$, contradicting $w' \notin S_0$. Hence it follows that $w'_{j-1} = r$ and $j-1 > r-t$, contradicting the selection of w. It follows that S is a cycle transversal.

It remains to bound $|S|$. Note that $|A| = (t+1) + (2t+1) = 3t+2$. Since each permutation in S_0 puts r or $r-1$ in a position indexed by $1, r-t, r$ and the other in a position indexed by A, we have that $|S_0| \le 6 \cdot |A| \cdot (r-2)! \le 6(3t+2)(r-2)!$.

Next, we bound S_1 and S_2. Let p be the probability that $\pi_1 \preceq \pi_2$ when permutations π_1 and π_2 of $[t]$ are chosen uniformly and independently at random. Conditioning on whether or not π_1 and π_2 are distinct gives $p = \frac{1}{2} \cdot (1 - \frac{1}{t!}) + 1 \cdot \frac{1}{t!} = \frac{1}{2} + \frac{1}{2(t!)}$.

Choose $u \in V(\mathrm{PSG}_r)$ with $u = (u_1,\dots,u_r)$ uniformly at random. We bound $\Pr(u \in S_1)$ by conditioning on the event $i \in A$, where i is the index such that $u_i = r-1$. Note that $\Pr(u \in S_1 \mid i \in A) = 0$, since $u \in S_1$ requires that $u_{r-t} = r$ and that $u \notin S_0$. Also, we have $\Pr(u \in S_1 \mid i \notin A) = \frac{1}{r-1} \cdot p$ since then $u_{r-t} = r$ with probability $1/(r-1)$, and, conditioned on $i \notin A$, the $(r-1)$ and r clusters in u are disjoint and so $\lambda_{r-1}(u)$ and $\lambda_r(u)$ are independently and uniformly distributed among all permutations of $[t]$. Hence $\Pr(u \in S_1) = \Pr(u \in S_1 | i \in A) \cdot \Pr(i \in A) + \Pr(u \in S_1 | i \notin A) \cdot \Pr(i \notin A) \le 0 + \frac{p}{r-1} \cdot \frac{r-|A|}{r} < \frac{p}{r}$. A similar computation shows that $\Pr(u \in S_2) < \frac{p}{r}$. Hence $|S| < r!(\frac{6(3t+2)}{r(r-1)} + \frac{2p}{r}) \le r!(\frac{6(3t+2)}{r(r-1)} + \frac{1}{r} \cdot (1 + \frac{1}{t!}))$. When $t = O(\log r / \log\log r)$, we obtain $|S| \le r! \left(\frac{1}{r} + O\left(\frac{\log r}{r^2 \log\log r}\right)\right)$. □

To obtain lower bounds on $\tau(\mathrm{PSG}_r)$, we find disjoint cycles. Given a tuple $(a_1, \ldots, a_r)$, the *forward shift operation* produces the tuple $(a_2, \ldots, a_r, a_1)$, and the *backward shift operation* produces $(a_r, a_1, \ldots, a_{r-1})$. A tuple is a *cyclic shift* of another if it can be obtained by a sequence of zero or more shift operations. Note that partitioning $V(\mathrm{PSG}_r)$ into the equivalence classes of the cyclic shift relation gives the family $\mathcal{F}$ of *shift cycles*, where $|\mathcal{F}| = r!/r$ and each $C \in \mathcal{F}$ has size r. Since every cycle transversal includes at least one vertex from each of these cycles, $\tau(\mathrm{PSG}_r) \ge |\mathcal{F}| = r!(1/r)$. We obtain a slight improvement by replacing certain cycles in $\mathcal{F}$ with a pair of cycles.

Lemma 9. *Let $u, v \in V(\mathrm{PSG}_r)$. Let u' be the vertex obtained from u by applying the forward shift operation, and let v' be the vertex obtained from v by applying the backward shift operation. We have $uv \in E(\mathrm{PSG}_r)$ if and only if $v'u' \in E(\mathrm{PSG}_r)$.*

Proof. Let $u = (u_1, \ldots, u_r)$ and $v = (v_1, \ldots, v_r)$. We have $u' = (u_2, \ldots, u_r, u_1)$ and $v' = (v_r, v_1, \ldots, v_{r-1})$. We have that $uv \in E(\mathrm{PSG}_r)$ if and only if $\rho(u_2, \ldots, u_r) = \rho(v_1, \ldots, v_{r-1})$ if and only if $v'u' \in E(\mathrm{PSG}_r)$. □

Lemma 10. *Let $r \ge 2$ and let C be a shift cycle in PSG_r. If C has a chord or loop, then $V(C)$ can be partitioned into two sets that contain cycles.*

Proof. Suppose that $C = \langle u_0, \ldots, u_{r-1} \rangle$ and that $u_i u_j \in E(\mathrm{PSG}_r)$ with $j \not\equiv i+1 \pmod r$. By Lemma 9, we have that $u_{j-1} u_{i+1} \in E(\mathrm{PSG}_r)$ (subscript arithmetic modulo r). Following C from u_j to u_i and traversing $u_i u_j$ completes a cycle C_1. Since $j \not\equiv i+1 \pmod r$, we have that $V(C_1)$ is a proper subset of $V(C)$, and in particular $u_{i+1} \notin V(C_1)$. Following C from u_{i+1} to u_{j-1} and traversing $u_{j-1} u_{i+1}$ completes a second cycle C_2 that is disjoint from C_1. □

Lemma 11. *The number of shift cycles in PSG_r that contain chords or loops equals $\varphi(r)$, where $\varphi(r)$ is the Euler Totient Function given by $\varphi(r) = |\{d \in [r] \colon \gcd(d, r) = 1\}|$.*

Proof. Let $d \in [r]$ be relatively prime to r, and let $x \in V(\mathrm{PSG}_r)$ be the vertex such that $x = (x_0, \ldots, x_{r-1})$ with $x_s \equiv ds \pmod r$. Let u_j be the vertex in PSG_r obtained from x by applying the forward shift operation j times, and let C be the shift cycle $\langle u_0, \ldots, u_{r-1} \rangle$. Let $u_j = (u_{j,0}, \ldots, u_{j,r-1})$ and observe that $u_{j,s} = u_{0,s+j} = x_{s+j}$. Let d' be the inverse of d modulo r, and note that $x_0 = r$ and $x_{d'} = 1$. We claim that $u_{d'} u_1 \in E(\mathrm{PSG}_r)$. Indeed, we have $u_{d',s} = x_{d'+s} \equiv d(d'+s) \equiv 1 + ds \pmod r$ and $u_{1,s-1} = x_s \equiv ds \pmod r$. It follows that $u_{d',0} \equiv 1 \pmod r$ implying $u_{d',0} = 1$, and similarly $u_{1,r-1} \equiv 0 \pmod r$ implying $u_{1,r-1} = r$. Also, we have that $u_{d',s} - 1 \equiv u_{1,s-1} \pmod r$. Since $2 \le u_{d',s} \le r$ and $1 \le u_{1,s-1} \le r-1$ for $1 \le s \le r-1$, it follows that $u_{d',s} - 1 = u_{1,s-1}$ for $1 \le s \le r-1$. Since the $(r-1)$-suffix of $u_{d'}$ matches the pattern of the $(r-1)$-prefix of u_1, we have that $u_{d'} u_1 \in E(\mathrm{PSG}_r)$ as claimed. Note that $u_{d'} u_1$ is a chord or a loop, since $1 - d' \equiv 1 \pmod r$ if and only if $d' \equiv 0 \pmod r$, which is impossible since d' is the inverse of d. Allowing d to range over the integers in $[r]$ that are relatively prime to r produces distinct cycles; indeed, given such a cycle C, we have that d is the value that follows r in each vertex of C.

Conversely, let C be a shift cycle in PSG_r that contains a chord or a loop. Let $C = \langle u_0, \ldots, u_{r-1} \rangle$ with $u_j = (u_{j,0}, \ldots, u_{j,r-1})$ for each j. We may assume the vertices of C are indexed so that $u_{0,0} = r$. Let $x = u_0$ with $x = (x_0, \ldots, x_{r-1})$. As above, we have $u_{j,s} = x_{j+s}$. Let $u_i u_j \in E(\mathrm{PSG}_r)$ with $j - i \not\equiv 1 \pmod r$. The $(r-1)$-suffix of u_i is $(u_{i,1}, \ldots, u_{i,r-1})$ or equivalently $(x_{i+1}, \ldots, x_{i+r-1})$; this is the list σ_1 obtained from the natural cyclic order on x by deleting x_i and recording the remaining $r-1$ elements in order. Similarly, the $(r-1)$-prefix of u_j is $(u_{j,0}, \ldots, u_{j,r-2})$ or equivalently $(x_j, \ldots, x_{j+r-2})$; this is the list σ_2 obtained from the natural cyclic order on x by deleting x_{j-1} and recording the remaining elements in order. Note that x_i and x_{j-1} are distinct entries of x. Since $u_i u_j \in E(\mathrm{PSG}_r)$, we have that σ_1 and σ_2 are both lists of size $r-1$ that pattern-match. Note that 1 cannot appear in both σ_1 and σ_2, since then the minimum element 1 would appear at distinct indices in σ_1 and σ_2. Similarly, r cannot appear in both σ_1 and σ_2. It follows that $\{x_i, x_{j-1}\} = \{1, r\}$. We may assume without loss of generality that $x_i = 1$ and $x_{j-1} = r$ (implying $j = 1$ as $x_0 = u_{0,0} = r$), since otherwise we may apply the same argument to $u_{j-1}u_{i+1} \in E(\mathrm{PSG}_r)$, which interchanges i and $j-1$.

Note that $\sigma_1 = (x_{i+1}, \ldots, x_{i+r-1})$ and σ_1 is a permutation of $\{2, \ldots, r\}$ since $x_i = 1$. Similarly, $\sigma_2 = (x_j, \ldots, x_{j+r-2}) = (x_1, \ldots, x_{r-1})$ and σ_2 is a permutation of $\{1, \ldots, r-1\}$ since $x_0 = r$. Since σ_1 and σ_2 pattern match, we have that $x_{i+t} - 1 = x_t$ for $1 \le t \le r-1$. Also, when $t = 0$, we have $x_{i+0} - 1 = x_i - 1 = 0$ and $x_0 = r$, implying that $x_{t+i} \equiv x_t + 1 \pmod r$ for all t. Iterating this congruence gives $x_{\ell i} \equiv x_{(\ell-1)i} + 1 \equiv \cdots \equiv x_0 + \ell \equiv r + \ell \equiv \ell \pmod r$. Since ℓ ranges over all congruence classes modulo r, so must ℓi. It follows that i and r are relatively prime; let d be the inverse of i modulo r. Setting $\ell = sd$, we have that s ranges over all congruence classes with ℓ and so $x_{\ell i} \equiv \ell \pmod r$ becomes $x_s \equiv sd \pmod r$. It follows that C is one of the $\varphi(r)$ cycles constructed above. □

Combining these results, we obtain the following.

Theorem 12. *Let $r \ge 2$. We have $\tau(\mathrm{PSG}_r) \ge r!(\frac{1}{r} + \frac{\varphi(r)}{r!})$.*

Proof. Let $\mathcal{F}$ be the family of shift cycles in PSG_r and note that $|\mathcal{F}| = r!/r$. By Lemma 11, we have that $\varphi(r)$ cycles in $\mathcal{F}$ have chords or loops, and by Lemma 10, each of these can be replaced with a pair of cycles, giving a family of $r!/r + \varphi(r)$ disjoint cycles in PSG_r. □

Corollary 13. *Let $r \ge 3$. We have $1 - \frac{1}{r} - O(\frac{\log r}{r^2 \log\log r}) \le a(\mathrm{PSG}_r)/r! \le 1 - \frac{1}{r} - \frac{\varphi(r)}{r!}$. Hence, the threshold for growing paths in r-uniform tournaments is in the range $\left[\left(1 - \frac{1}{r} - O(\frac{\log r}{r^2 \log\log r})\right) r!,\ \left(1 - \frac{1}{r} - \frac{\varphi(r)-1}{r!}\right) r!\right]$.*

Proof. Recall that $\tau(\mathrm{PSG}_r) + a(\mathrm{PSG}_r) = |V(\mathrm{PSG}_r)| = r!$. The bounds on $a(\mathrm{PSG}_r)$ arise by taking $t = \Theta(\log r/\log\log r)$ in Theorem 8, and by Theorem 6, the threshold for growing paths equals $1 + a(\mathrm{PSG}_r)$. □

2.3 Linear Path Threshold

In this section, we show that when $k \ge (1 - \frac{1}{r} + \frac{1}{r!})r!$, an (r,k)-tournament has paths of linear size. The following proposition is well-known. We include the short proof for completeness.

Proposition 14. *Let G be an r-graph with average degree d. We have that G has a subgraph with minimum degree at least d/r. Moreover, if $r \ge 2$ and $d > 0$, then G contains a subgraph with minimum degree larger than d/r.*

Proof. If $d = 0$, the claim is trivial. If $r = 1$, then a vertex of maximum degree induces a subgraph with minimum degree at least d. So assume $r \ge 2$ and $d > 0$.

Let G_0 be the smallest subgraph of G with average degree at least d. Let $n = |V(G_0)|$ let $m = |E(G_0)|$. Since $d > 0$ and $r \ge 2$, it follows that G_0 has an edge and so $n \ge r \ge 2$. Note that G_0 has average degree rm/n, with $rm/n \ge d$. Let $k = \delta(G_0)$ and let u be a vertex in G_0 with degree k. Note that $G_0 - u$ has $m - k$ edges and $n - 1$ vertices, so that $G_0 - u$ has average degree $r(m-k)/(n-1)$. By extremality of G_0, we have $r(m-k)/(n-1) < d \le rm/n$. It follows that $k > m/n \ge d/r$. □

Our next lemma shows that r-digraphs without long paths must contain large subgraphs without closed walks.

Lemma 15. *Let G be a fully directed n-vertex r-graph and suppose that $P_{s+1}^{(r)} \not\subseteq G$. There is an induced subgraph H such that $|V(H)| \ge (1 - o(1))(n/(rs))^{1/(r-1)}$, where the $o(1)$ term depends only on n and r and, for each fixed r, approaches 0 as $n \to \infty$, and every walk in H has size at most $\max\{r-1, s\}$. In particular, H has no closed walks or cycles.*

Proof. For each ordered $(r-1)$-tuple of vertices σ, let P_σ be a maximum path in G whose last $r-1$ vertices are as listed in σ. Let $s_\sigma = |V(P_\sigma)|$. Note that for each σ, we have $r - 1 \le s_\sigma \le s$. An ordered r-tuple π is *good* if $w \in V(P_\sigma)$ where the first $r-1$ entries in π are σ and the last entry in π is w. An unordered r-set S is *good* if at least one of its $r!$ permutations is good.

Let m be the number of good (unordered) r-sets in G. Since $\sum_\sigma |V(P_\sigma)| \ge m$, it follows that $|V(P_\sigma)| \ge m/n^{r-1}$ for some $(r-1)$-tuple σ. Therefore $m \le sn^{r-1}$. Let T be a maximum set of vertices in G that contains no good r-set, and let $t = |T|$. Since every $(t+1)$-set of vertices contains a good r-set, it follows from the de Caen bound [3] on the Turán number of $K_{t+1}^{(r)}$ that $m \ge (1 - o(1))\frac{\binom{n}{r}}{\binom{t}{r-1}}$. Therefore

$(1-o(1))\frac{\binom{n}{r}}{\binom{t}{r-1}} \le m \le sn^{r-1}$, implying $(1-o(1))\frac{n_{(r)}}{rt_{(r-1)}} \le sn^{r-1}$. It follows that $(1-o(1))\left(\frac{n^r}{rs}\right)\left(\frac{n_{(r)}}{n^r}\right) \le n^{r-1}t_{(r-1)} \le n^{r-1}t^{r-1}$, and so $t \ge \left[(1-o(1))\left(\frac{n_{(r)}}{n^r}\right)\left(\frac{n}{rs}\right)\right]^{1/(r-1)} = (1-o(1))\left(\frac{n}{rs}\right)^{1/(r-1)}$.

Let $H = G[T]$. We show that every walk in H has size at most s. Note that if $\pi \in E(H)$, then the underlying set of π is not good, and so $w \notin V(P_\sigma)$ where π consists of σ followed by w. Therefore w extends a maximum path in G ending in σ to a longer one in G ending in σ', where σ' is the last $r-1$ vertices in π. It follows that $s_{\sigma'} \ge s_\sigma + 1$. Let W be a walk in H, with $W = u_1 \dots u_\ell$. If $\ell \le r-1$, then the claim holds. Otherwise $\ell \ge r$ and if σ is the $(r-1)$-interval of W ending at u_j, then $s_\sigma \ge j$. In particular, when σ ends at u_ℓ, we have $\ell \le s_\sigma \le s$. □

For $n \ge r$, the *tight cycle* $C_n^{(r)}$ is the n-vertex r-digraph whose vertices are arranged cyclically and whose edges are the intervals of size r. Similarly to paths, all cycles considered in this paper are tight unless otherwise specified. Using Proposition 14, we obtain long paths in a directed r-graph with many copies of $C_r^{(r)}$.

Lemma 16. *Let G be an n-vertex r-digraph. If G contains m copies of the r-vertex cycle $C_r^{(r)}$, then G contains a path on at least $m/n_{(r-1)} + (r-1)$ vertices.*

Proof. Let $\mathcal{C}$ be the set of copies of $C_r^{(r)}$ in G. For each $C \in \mathcal{C}$, let $\psi(C)$ be the set of $(r-1)$-intervals in the cyclic ordering of $V(C)$. Since each vertex in C starts an $(r-1)$-interval, we have that $|\psi(C)| = r$. Let H be the auxiliary r-graph whose vertices are the $(r-1)$-tuples of $V(G)$ with edge set $\{\psi(C)\colon C \in \mathcal{C}\}$. Note that $|E(H)| = m$ and $|V(H)| = n_{(r-1)}$, and so H has average degree $mr/n_{(r-1)}$. Since H is an r-graph with average degree $mr/n_{(r-1)}$, it follows that H has a subgraph H_0 with minimum degree at least $m/n_{(r-1)}$. Let t be the minimum degree of H_0.

Let P be a longest path in G with an $(r-1)$-suffix in $V(H_0)$, and let $(u_1, \dots, u_{r-1})$ be this suffix. Since H_0 has minimum degree t, it follows that there exist distinct vertices $v_1, \dots, v_t \in V(G)$ such that $C_j \in \mathcal{C}$ where $C_j = \langle u_1 \dots u_{r-1} v_j\rangle$ and $\psi(C_j) \in E(H_0)$. In particular, both $(u_1, \dots, u_{r-1})$ and $(u_2, \dots, u_{r-1}, v_j)$ are vertices in the edge $\psi(C_j)$ in H_0. It follows that $v_j \in V(P)$, or else appending v_j to P gives a longer path ending at $(u_2, \dots, u_{r-1}, v_j) \in V(H_0)$. Note that $|V(P)| \ge (r-1) + t$. □

We pause to discuss the sharpness of Lemma 16 in the case that $m = \binom{n}{r}$. When $m = \binom{n}{r}$, Lemma 16 implies that G has a path of size at least $n/(r!)$. This is sharp up to a factor that is polynomial in r.

Theorem 17. *For each positive n and r, there is an n-vertex r-digraph G such that every r-set contains a copy of $C_r^{(r)}$ and every path in G has size at most $r + 2r^4(n/(r!))$.*

Proof. Let F be the digraph whose vertices are the copies of $C_r^{(r)}$ in the complete r-digraph on $[r]$, with an edge from C to C' if and only if $ee' \in E(\mathrm{PSG}_r)$ for some $e \in E(C)$ and some $e' \in E(C')$. Note that F is the digraph obtained from PSG_r by contracting the equivalence classes of cyclic shifts and discarding loops and parallel edges. Since the equivalence classes are disjoint and have size r, it follows that $|V(F)| = |V(\mathrm{PSG}_r)|/r = (r-1)!$. Let $x \in \mathrm{PSG}_r$. Note that $N^+(x)$ consists of the cyclic shift x' of x and $r-1$ other outneighbors. When the r cyclic shifts of x are contracted to form a vertex C in F, each contributes at most $r-1$ to the outdegree of C. It follows that $d_F^+(C) \le r(r-1)$. Similarly, $d_F^-(C) \le r(r-1)$. Let F' be the underlying graph of F and note that $\Delta(F') \le 2r(r-1) < 2r^2$. It follows that F' has an independent set I with $|I| \ge |V(F')|/(1+\Delta(F)) \ge (r-1)!/(2r^2)$.

Let G be the r-digraph on $[n]$ as follows. Let $t = |I|$, let $I = \{C_1, \dots, C_t\}$ and let $\{X_1, \dots, X_t\}$ be an equipartition of $[n]$ into t parts. For each $S \in \binom{[n]}{r}$, let $\psi(S)$ be the index j of the part X_j containing $\min S$. For convenience, we extend ψ in the natural way to r-tuples of distinct vertices. For each S, we include in G the r-cycle on S whose edges pattern-match the edges in $C_{\psi(S)}$, so that $(v_1, \dots, v_r) \in E(G)$ if and only if $\rho(v_1, \dots, v_r) \in E(C_j)$ where ρ is the canonical pattern function and $j = \psi(v_1, \dots, v_r)$.

Suppose that $u_1 \dots u_s$ is a path in G. We claim that ψ is constant on the r-subintervals of $u_1 \dots u_s$. Let e and e' be consecutive r-subintervals of $u_1, \dots, u_s$ with e before e'. Since the $(r-1)$-suffix of e equals the $(r-1)$-prefix of e', it follows that $\rho(e)\rho(e') \in E(\mathrm{PSG}_r)$. Also, since $e, e' \in E(G)$, it follows that $\rho(e) \in E(C_j)$ and $\rho(e') \in E(C_{j'})$ where $j = \psi(e)$ and $j' = \psi(e')$. If $j \ne j'$, then F has an edge from C_j to $C_{j'}$, but this contradicts that C_j and $C_{j'}$ are both members of the independent set I in the underlying graph F'. It follows that ψ is constant on the r-subintervals of $u_1 \dots u_s$.

Let j be the common value that ψ assigns to each r-subinterval of $u_1 \dots u_s$. It follows that each r-subinterval of $u_1 \dots u_s$ contains at least one vertex in X_j. Therefore $s \le r|X_j| \le r\lceil n/|I|\rceil \le r + 2r^4 \cdot n/(r!)$. □

Our next theorem establishes an upper bound on the linear path threshold.

Theorem 18. *If $k = (1 - \frac{1}{r} + \frac{1}{r!})r!$, then $f(n,r,k) \geq n/r!$. It follows that the linear path threshold is at most $(1 - \frac{1}{r} + \frac{1}{r!})r!$*

Proof. Let G be an n-vertex (r,k)-tournament. Since each r-set omits fewer than $(r-1)!$ edges, it follows that each r-set has a copy of $C_r^{(r)}$. Applying Lemma 16 with $m = \binom{n}{r}$, we obtain a path P in G such that $|V(P)| \geq \frac{\binom{n}{r}}{n_{(r-1)}} + r - 1 = \frac{n-r+1}{r!} + r - 1 \geq n/r!$. $\square$

2.4 Spanning Path Threshold

Using the Lovász Local Lemma, it is not difficult to show that if $k \geq (1 - \frac{1}{e(2r-1)})r!$, then $f(n,r,k) = n$. In this section, we relax the hypothesis to $k > (1 - \frac{1}{4(r-1)})r!$.

Proposition 19. *Let X be a random variable such that $X \leq M$. If $\alpha < M$, then $\Pr(X \geq \alpha) \geq (\mathbb{E}(X) - \alpha)/(M - \alpha)$.*

Proof. Applying Markov's inequality to the non-negative random variable $M - X$, we have $\Pr(X \geq \alpha) = \Pr(M - X \leq M - \alpha) = 1 - \Pr(M - X > M - \alpha) \geq 1 - \frac{\mathbb{E}(M-X)}{M-\alpha} = (\mathbb{E}(X) - \alpha)/(M - \alpha)$. $\square$

Proposition 20. *Let G be an n-vertex fully directed r-graph with $n \geq r$, and let $t = \min\{n, 2r-2\}$. If the edge density $|E(G)|/n_{(r)}$ of G is greater than $1 - \frac{1}{4(r-1)}$, then more than half of the $(r-1)$-tuples of distinct vertices of G begin more than $\frac{1}{2}n_{(t)}/n_{(r-1)}$ paths of size t.*

Proof. Every r-tuple of distinct vertices in G appears consecutively in $\binom{n-r}{t-r}(t-r+1)!$ of the t-tuples of distinct vertices. Since G has edge density more than $1 - \frac{1}{4(r-1)}$, it follows that G has fewer than $\frac{1}{4(r-1)}n_{(r)}$ non-edges, and so fewer than $\frac{1}{4(r-1)}n_{(r)}\binom{n-r}{t-r}(t-r+1)!$ of the t-tuples of distinct vertices in G have a non-edge as a substring. Since $t \leq 2r-2$, this is at most $\frac{1}{4}n_{(t)}$. So G has more than $\frac{3}{4}n_{(t)}$ paths of size t.

Let σ be an $(r-1)$-tuple of distinct vertices in G chosen uniformly at random, let X be the number of paths in G of size t that begin with σ, and let $M = (n-(r-1))_{(t-(r-1))} = n_{(t)}/n_{(r-1)}$. Note that $X \leq M$ for each σ, and directly applying the definition of expectation gives $\mathbb{E}(X) = \frac{1}{n_{(r-1)}}\sum_\sigma X(\sigma) > \frac{1}{n_{(r-1)}} \cdot \frac{3}{4}n_{(t)} = \frac{3}{4}M$. Let $\varepsilon = \mathbb{E}(X) - \frac{3}{4}M$. By Proposition 19, we have $\Pr(X > M/2) \geq \Pr(X \geq M/2 + \varepsilon) \geq \frac{\mathbb{E}(X) - (M/2+\varepsilon)}{M - (M/2+\varepsilon)} = \frac{M/4}{M/2 - \varepsilon} > 1/2$. $\square$

Theorem 21. *Let $n \geq r \geq 2$. If $k > (1 - \frac{1}{4(r-1)})r!$ then $f(n,r,k) = n$.*

Proof. Let G be an n-vertex (r,k)-tournament with $k > (1 - \frac{1}{4(r-1)})r!$. We show that G has a spanning path. A path P is *flexible* if more than half of the $(r-1)$-tuples of distinct vertices in $V(G) - V(P)$ form a path when appended to P. We show that G has a flexible path of size 1. If u is an r-tuple of distinct vertices chosen uniformly at random and $u = (u_1, \ldots, u_r)$, then $k/r! = \Pr(u \in E(G)) = \sum_{v \in V(G)} \Pr(u \in E(G) | u_1 = v) \cdot \Pr(u_1 = v)$. It follows that some vertex $v \in V(G)$ begins an edge with at least a $k/r!$ fraction of the $(r-1)$-tuples of distinct vertices in $G - v$. Since $k/r! > 1/2$, it follows that such a vertex forms a flexible path of size 1.

Let P be a maximal flexible path. Let $G' = G - V(P)$, let $n' = |V(G')|$, and note that $n' \geq r-1$ since flexibility requires that at least one $(r-1)$-tuple of distinct vertices in G' extends P. If $n' = r-1$, then P extends to a spanning path in G. So we may assume $n' \geq r$. Let $t = \min(n', 2r-2)$. Since G' is an (r,k)-tournament with $k > (1 - \frac{1}{4(r-1)})r!$, Proposition 20 implies that the set A of $(r-1)$-tuples of distinct vertices in G' which begin more than $\frac{1}{2}n'_{(t)}/n'_{(r-1)}$ paths of size t has size more than $\frac{1}{2}n'_{(r-1)}$.

Suppose that $n' \geq 2r-2$, and note that $t = 2r-2$ in this case. Since P is flexible, the set B of $(r-1)$-tuples of distinct vertices in G' which form paths when appended to P has size more than $\frac{1}{2}n'_{(r-1)}$. By the pigeonhole principle, there exists $(x_1, \ldots, x_{r-1}) \in A \cap B$. Let Q be the path in G obtained by appending $x_1 \ldots x_{r-1}$ to P, let $G'' = G - V(Q)$, and let $n'' = V(G'')$. Since $(x_1, \ldots, x_{r-1}) \in A$, there are more than $\frac{1}{2}n'_{(t)}/n'_{(r-1)}$ paths in G' of size t that have $x_1 \ldots x_{r-1}$ as a prefix. Since $n'_{(t)}/n'_{(r-1)} = (n'-(r-1))_{(t-(r-1))} = (n'-(r-1))_{(r-1)} = n''_{(r-1)}$, it follows that Q is a larger flexible path, contradicting the maximality of P.

Therefore $r \leq n' < 2r-2$. In this case $t = n'$ and so Proposition 20 gives us that over half the $(r-1)$-tuples of distinct vertices of G' extend to spanning paths of G'. Since P is flexible, at least one of these $(r-1)$-tuples extends P. So P extends to a spanning path of G. $\square$

3 Small Uniformity

In this section, we present arguments that apply to (r, k)-tournaments when r is small. Most of our work here is restricted to the case $r = 3$ with a brief discussion of the known bounds on thresholds when $r \in \{4, 5\}$. We begin with the case $r = 3$.

3.1 Paths in 3-uniform tournaments

When $r = 3$, Proposition 1 implies $f(n, 3, 2) \leq 3$, Theorem 21 gives the trivial result that $f(n, 3, 6) = n$, and Theorem 18 implies that $f(n, 3, 5) = \Theta(n)$. In this section, we obtain improved results when $r = 3$ and $k \in \{3, 4, 5\}$.

3.1.1 Paths in $(3, 5)$-tournaments

We show that every $(3, 5)$-tournament has a spanning path.

Proposition 22. *$f(n, 3, 5) = n$.*

Proof. Suppose for a contradiction that the identity does not hold, and let n be the smallest integer such that $f(n, 3, 5) < n$. Let G be an n-vertex $(3, 5)$-tournament without a spanning path. Let u be a vertex in G, and let P be a spanning path of $G - u$, where $P = v_1 v_2 \cdots v_{n-1}$.

Suppose there exists an integer i such that $u v_i v_{i+1} \in E(G)$, and let i be the least such integer. Let $Q = v_1 \cdots v_{i-1} u v_i v_{i+1} \cdots v_{n-1}$. We claim that Q is a spanning path in G. Let π be a 3-interval in Q. If u is not an entry in π, then $\pi \in E(P) \subseteq E(G)$. Otherwise consider the position of u in π. If u is in the first position, then $\pi = u v_i v_{i+1} \in E(G)$ by the choice of i. If u is in the second position, then $\pi = v_{i-1} u v_i$. By minimality of i, we have that $u v_{i-1} v_i \notin E(G)$. Since G is a $(3, 5)$-tournament, it follows that all other permutations of $\{u, v_{i-1}, v_i\}$ are edges in G, including π. The case where u is in the third position is similar.

Otherwise, there is no such integer i and we claim that Q is a spanning path, where $Q = v_1 \cdots v_{n-1} u$. Indeed, if π is a 3-interval containing u, then $\pi = v_{n-2} v_{n-1} u$. Since $u v_{n-2} v_{n-1} \notin E(G)$ and G is a $(3, 5)$-tournament, it follows that $\pi \in E(G)$. $\square$

The argument in Proposition 22 easily extends to show that $f(n, r, r! - 1) = n$ for $r \geq 2$, but Theorem 21 already implies this when $r \geq 4$, so we present the argument only in the case $r = 3$.

3.1.2 Paths in $(3, 4)$-tournaments

In this section, we show that every n-vertex $(3, 4)$-tournament has a path of size $\Omega(n^{1/5})$. However, significantly stronger results may be possible.

Conjecture 23. *Every $(3, 4)$-tournament has a spanning path. That is, $f(n, 3, 4) = n$.*

In support of this conjecture, we note that Misha Lavrov [6] used computer search to verify Conjecture 23 when $n \leq 7$ and obtained some evidence to suggest that perhaps Conjecture 23 may be further strengthened to assert that every n-vertex $(3, 4)$-tournament has at least 2^{n-1} spanning paths. This would be best possible. Let G be the $(3, 4)$-tournament $[n]$ with $(u, v, w) \in E(G)$ if and only if $\max\{u, v, w\} \neq v$. Every spanning path in G consists of a spanning path of $G - n$ with n added to the beginning or the end, and so G has twice the number of spanning paths as $G - n$. It follows by induction that G has 2^{n-1} spanning paths.

To prove our lower bound on $f(n, 3, 4)$, we first use Lemma 15 to reduce to the case that G has no triangles, and then we show that in such tournaments the longest paths are pairwise intersecting. Note that if G is a $(3, 4)$-tournament with no triangles, then for each triple $\{u, v, w\}$ it must be that G contains exactly two edges in $\{uvw, vwu, wuv\}$ and two edges in $\{wvu, vuw, uwv\}$.

Lemma 24. *If G is an n-vertex $(3, 4)$-tournament with no triangles, then the family of maximum paths in G is pairwise intersecting.*

Proof. Suppose for a contradiction that A and B are disjoint maximum paths in G, with $A = a_1 \ldots a_t$ and $B = b_1 \ldots b_t$. An *out edge* of A is an edge of the form (a_j, a_{j+1}, b_j) or (a_j, a_{j+1}, b_{j+1}), and an *out edge* of B is an edge of the form (b_j, b_{j+1}, a_j) or (b_j, b_{j+1}, a_{j+1}).

Suppose G has out edges. Let $(a_j, a_{j+1}, b_{j'})$ or $(b_j, b_{j+1}, a_{j'})$ be an out edge chosen first to maximize j, and next to maximize $j' \in \{j, j+1\}$. Without loss of generality, we may assume the chosen out edge is $(a_j, a_{j+1}, b_{j'})$. Let $Q = a_1 \cdots a_{j+1} b_{j'} \cdots b_t$. We claim that Q is a path. Clearly, $(a_j, a_{j+1}, b_{j'}) \in E(G)$.

If $j' < t$, then $(a_{j+1}, b_{j'}, b_{j'+1}) \in E(G)$ since $(b_{j'}, b_{j'+1}, a_{j+1}) \notin E(G)$ by our extremal choice of an out edge, and since G is a $(3,4)$-tournament with no tight cycle, this forces $(b_{j'+1}, a_{j+1}, b_{j'}), (a_{j+1}, b_{j'}, b_{j'+1}) \in E(G)$. All other consecutive triples in Q are edges in A or B. Since Q is a tight path in G with at least $t+1$ vertices, we obtain a contradiction.

It follows that G has no out edges. Let $Q = a_t b_t a_{t-1} b_{t-1} \cdots a_2 b_2 a_1 b_1$. We claim that Q is a path in G. Indeed, if $(a_j, b_j, a_{j-1}) \notin E(G)$, then since G is a $(3,4)$-tournament with no triangles, this would force $(b_j, a_{j-1}, a_j), (a_{j-1}, a_j, b_j) \in E(G)$, but the latter is an out edge of A. A similar argument shows that $(b_j, a_{j-1}, b_{j-1}) \in E(G)$. Since Q has more than t vertices, we again obtain a contradiction. □

Lemma 25. *Let $s \geq 1$ and let G be an s^2-vertex $(3,4)$-tournament with no triangles. We have that $P_s^{(3)} \subseteq G$.*

Proof. By induction on s. The claim is clear when $s = 1$. Suppose that $s \geq 2$. By induction, G contains an $(s-1)$-vertex path Q. Let $G' = G - V(Q)$, and note that $|V(G')| = s^2 - (s-1) \geq (s-1)^2$. So G' also contains an $(s-1)$-vertex path Q'. Since Q and Q' are disjoint, these cannot both be maximum paths by Lemma 24. Hence G contains a path on s vertices. □

We are now able to give our lower bound on $f(n,3,4)$.

Theorem 26. $f(n,3,4) \geq (1 - o(1))(n/3)^{1/5}$.

Proof. Let G be a $(3,4)$-tournament, and let s be the maximum integer such that $P_s^{(3)} \subseteq G$. By Lemma 15, G contains an induced acyclic subgraph H on at least $(1 - o(1))\sqrt{n/(3s)}$ vertices. By Lemma 25, H contains a path on $(1 - o(1))(n/(3s))^{1/4}$ vertices. Hence $f(n,3,4) \geq \max\{s, (1 - o(1))(n/(3s))^{1/4}\} \geq (1 - o(1))(n/3)^{1/5}$. □

3.1.3 Paths in $(3,3)$-tournaments

Let X be a set of r-tuples. The *shift graph on X* is the directed graph with an edge from u to v if and only if the $(r-1)$-suffix of u equals the $(r-1)$-prefix of v.

Theorem 27. $f(n,3,3) \leq 2 \lg n + 6$.

Proof. Suppose that $n = 2^t$ for some integer t. We construct a $(3,3)$-tournament G on $\mathbb{F}_2^t$ as follows. Given a triple $e \in \binom{V(G)}{3}$, let $\alpha(e)$ be the minimum index ℓ such that the vertices in e are not all equal in the ℓ coordinate. Let $\beta(e)$ be the minimum index ℓ such that the pair in e that agrees at coordinate $\alpha(e)$ disagrees at coordinate ℓ. Note that $\beta(e) > \alpha(e)$. We extend α and β in the natural way to ordered triples.

Given an ordered triple a with $a = (u, v, w)$, $i = \alpha(a)$, and $j = \beta(a)$, we include a in $E(G)$ if the minor of the $(t \times 3)$-matrix with columns u, v, w induced by rows i and j matches one of the following *allowed patterns*.

$$P_1 : \begin{bmatrix} 0 & 1 & 0 \\ * & * & * \end{bmatrix} \qquad P_2 : \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & * \end{bmatrix} \qquad P_3 : \begin{bmatrix} 0 & 1 & 1 \\ * & 1 & 0 \end{bmatrix} \qquad P_4 : \begin{bmatrix} 1 & 1 & 0 \\ * & * & * \end{bmatrix}$$

In the above patterns, the asterisks match either character. We claim that G is a $(3,3)$-tournament. Let $e = \{u, v, w\}$, let $i = \alpha(e)$, and let $j = \beta(e)$. Without loss of generality, we may suppose that $u_i = v_i \neq w_i$, $u_j = 0$ and $v_j = 1$. If $u_i = v_i = 0$, then (u, w, v) and (v, w, u) match P_1, and (u, v, w) matches P_2. If $u_i = v_i = 1$, then (u, v, w) and (v, u, w) match P_4, and (w, v, u) matches P_3. So G is a $(3,3)$-tournament.

Suppose that Q is path, with $Q = x_1 \cdots x_m$. Let $a_s = (x_s, x_{s+1}, x_{s+2})$ for $1 \leq s \leq m-2$. We claim that if a_s matches the *trap pattern* $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & * \end{bmatrix}$ with $i = \alpha(a_s)$ and $j = \beta(a_s)$, then a_{s+1} also matches the trap pattern with $\alpha(a_{s+1}) < \beta(a_{s+1}) = i = \alpha(a_s)$. First, note that i is the minimum index such that x_{s+1} and x_{s+2} differ, implying $\alpha(a_{s+1}) \leq i$. For equality to hold, a_{s+1} must match a pattern whose top row begins with a 1 followed by a 0, and this is incompatible with all allowed patterns. Hence $\alpha(a_{s+1}) < i$, meaning that the first disagreement among vertices in a_{s+1} occurs at a coordinate $\alpha(a_{s+1})$ where x_{s+3} disagrees with the common value in x_{s+1} and x_{s+2}. Hence $\beta(a_{s+1})$ is the least coordinate where x_{s+1} and x_{s+2} disagree, giving $\beta(a_{s+1}) = i$. It follows that a_{s+1} matches $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & * \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & * \end{bmatrix}$. Since the first of these is incompatible with the allowed patterns, the claim follows.

Next, we claim that if $\alpha(a_s) \neq \alpha(a_{s+1})$, then either a_s matches P_2 or a_{s+1} matches the trap pattern. Suppose that $\alpha(a_s) \neq \alpha(a_{s+1})$ and a_s matches a pattern in $\{P_1, P_3, P_4\}$. We show that a_{s+1} matches the trap pattern. Let $i = \alpha(a_s)$, let $j = \beta(a_s)$, and let j' be the index of the first coordinate where x_{s+1} and x_{s+2} differ. Note that $i \leq j'$. If $i = j'$, then a_s does not match P_3 and so a_s matches a pattern in $\{P_1, P_4\}$. In this case, we have $x_{s+1}(j') = x_{s+1}(i) = 1$ and $x_{s+2}(j') = x_{s+2}(i) = 0$. Otherwise, $i < j'$, implying that that $x_{s+1}(i) = x_{s+2}(i)$. Therefore a_s matches P_3 and $j = j'$, again giving $x_{s+1}(j') = x_{s+1}(j) = 1$ and $x_{s+2}(j') = x_{s+2}(j) = 0$. It follows that $\alpha(a_{s+1}) < j'$, since $\alpha(a_{s+1}) = j'$ would require a_{s+1} to match a pattern whose first row begins with a 1 and then a 0, which is inconsistent with the allowable patterns. Since $\alpha(a_{s+1}) < j'$, it follows that x_{s+1} and x_{s+2} agree at coordinate $\alpha(a_{s+1})$ and therefore $\beta(a_{s+1}) = j'$. Hence a_{s+1} matches $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & * \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & * \end{bmatrix}$. The former is inconsistent with the allowable patterns, and so the claim follows.

A *block* is a maximal interval $[a_s, \dots, a_{s'}]$ of edges of Q such that $\alpha(a_s) = \cdots = \alpha(a_{s'})$. Partition $E(Q)$ into blocks $B_1, \dots, B_r$. Note that within each B_j, the edges match patterns that follow a walk in the shift graph on the top row of the patterns. Note that this shift graph is acyclic, and the maximum path has order 3. It follows that $|B_j| \leq 3$ for each j. We claim there is at most one index j such that the α component of edges in B_j is less than the α component of edges in B_{j+1}. Let j be the least index such that adjacent blocks B_j and B_{j+1} satisfy $\alpha(a_s) < \alpha(a_{s+1})$, where a_s is the last edge in B_j and a_{s+1} is the first edge in B_{j+1}. Since $\alpha(a_s) \neq \alpha(a_{s+1})$, either a_s matches P_2 or a_{s+1} matches the trap pattern. If a_s matches P_2, then since x_{s+1} and x_{s+2} differ in $\alpha(a_s)$, it follows that $\alpha(a_{s+1}) \leq \alpha(a_s)$, contradicting $\alpha(a_s) < \alpha(a_{s+1})$. Hence a_{s+1} matches the trap pattern and $\alpha(a_{s+1}) > \cdots > \alpha(a_{m-2})$. It follows that $r \leq 2t$.

Next, we claim that at most one block has size more than 1. Let i be the least integer such that $|B_i| > 1$. Note that if the last edge in B_i matches P_2, then $|B_i| = 1$ since in the successor graph, the top row of P_2 has indegree zero. Hence $|B_i| > 1$ implies that the last edge a_s in B_i does not match P_2 and so all edges in Q after a_s match the trap pattern, giving $|B_j| = 1$ for $j > i$. It follows that $|E(Q)| \leq (2t-1) + 3 = 2t + 2$ and $m = |V(Q)| = |E(Q)| + 2 \leq 2t + 4 = 2\lg n + 4$.

If n is not a power of two, then we apply the argument to the integer n', where n' is the least power of 2 greater than n. Since $n' \leq 2n$, it follows that $f(n,3,3) \leq f(n',3,3) \leq 2\lg n' + 4 \leq 2\lg n + 6$. □

Theorem 28. $f(n,3,3) \geq \Omega\left(\left(\frac{\ln n}{\ln\ln n}\right)^{1/2}\right)$.

Proof. Let G be a $(3,3)$-tournament on $[n]$, and let s be the size of a maximum path in G. By Lemma 15, there is a subtournament H with $|V(H)| \geq (1 - o(1))\sqrt{n/3s}$ such that every walk in H has size at most s. Let $n' = |V(H)|$; we may assume that $V(H) = \{1, \dots, n'\}$.

Let F be a copy of the complete graph on $\{1, \dots, n'\}$; we use H to color the edges of F as follows. For each pair of vertices u, v in H with $u < v$, let $\ell(uv) = (a, b)$ where a is the maximum length of a walk in H ending uv and b is the maximum length of a walk in H ending vu. We claim that there is no monochromatic triangle in F under the coloring ℓ. Suppose for a contradiction that uvw is a monochromatic triangle with $u < v < w$. Let (a, b) be the common color on uvw in F. Note that $uvw \notin E(H)$, since a walk of length a ending at uv would extend to a walk of length $a+1$ ending at vw, contradicting that $\ell(vw)$ has first component a. Similarly, $wvu \notin E(H)$ as both $\ell(vw)$ and $\ell(uv)$ have equal second component b. Since H is a $(3,3)$-tournament, it follows that $E(H)$ contains $\{vwu, wuv\}$ or $E(H)$ contains $\{uwv, vuw\}$.

Suppose that $\{vwu, wuv\} \subseteq E(H)$. Since a walk ending vw of length a extends to a walk ending wu of length $a+1$, it follows that $b \geq a+1$. Also, since $wuv \in E(H)$, a walk ending wu of length b extends to a walk ending uv of length $b+1$, and so $a \geq b+1$. This is a contradiction, and the case that $\{uwv, vuw\} \subseteq E(H)$ leads to a similar contradiction.

Recall that the p-color Ramsey number for triangles $R(3;p)$ satisfies $R(3;p) \leq 3p!$. Since walks in H have size at most s, the lengths range from 0 to $s - (r-1)$, and so ℓ gives a s^2-edge-coloring of a copy of $K_{n'}$ with no monochromatic triangles. It follows that $(1 - o(1))\sqrt{n/3s} \leq n' < R(3; s^2) \leq 3(s^2)! \leq 3(s^2)^{s^2} = 3s^{2s^2}$. Taking the natural log of both sides gives $(1/2)(\ln(n/(3s))) - o(1) \leq \ln(3) + 2s^2\ln(s)$ or $(1/2)(\ln n) - o(1) \leq (3/2)\ln(3) + (2s^2 + (1/2))\ln(s)$. If $s \geq \ln n$, then G has a path on $\ln n$ vertices and the bound follows. Otherwise $s < \ln n$ and it follows that $(1/2)\ln n - o(1) \leq (3/2)\ln(3) + (2s^2 + (1/2))\ln\ln n$ and so $\frac{\ln n}{2\ln\ln n} - o(1) \leq 2s^2 + (1/2) \leq 3s^2$. Hence $s \geq [\frac{\ln n}{6\ln\ln n} - o(1)]^{1/2}$. □

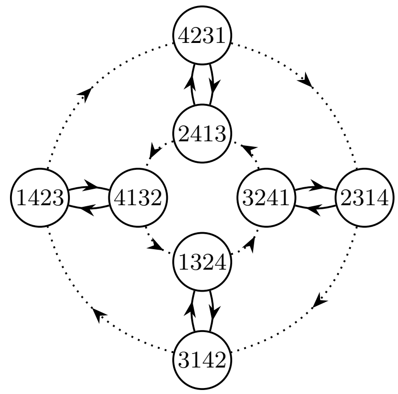

Figure 2: The two shift cycles above (dotted edges) in PSG_4 are replaced with four 2-cycles (solid edges) in our construction of a family of disjoint cycles.

3.2 Paths in r-uniform tournaments for $r \in \{4, 5\}$

In this section, we summarize known results for $r \in \{4, 5\}$ in Figure 3. Most of these are obtained by applying general theorems in Section 2. To obtain the threshold for growing paths when $r \in \{4, 5\}$, we find $\tau(\mathrm{PSG}_4)$ and $\tau(\mathrm{PSG}_5)$.

A list of integers $(u_1, \ldots, u_r)$ has a *bump* at position i if $u_i > u_{i-1}$ and $u_i > u_{i+1}$.

Proposition 29. *We have* $\tau(\mathrm{PSG}_4) = 10$ *and* $a(\mathrm{PSG}_4) = 4! - \tau(\mathrm{PSG}_4) = 14$.

Proof. Note that by Proposition 7, for $r \geq 3$, there is no edge uv in PSG_r with $u = (u_1, \ldots, u_r)$, $v = (v_1, \ldots, v_r)$, $u_r = r$, and $v_1 = r$.

We construct a cycle transversal S in PSG_4 of size 10. Let S consist of the vertices 1234, 4321, and the permutations (a_1, a_2, a_3, a_4) of $[4]$ such that $\max\{a_1, a_2, a_3\} = a_2$. Note that $|S| = 2 + (1/3) \cdot 4! = 10$. Let C be a cycle in PSG_4. Suppose for a contradiction that $V(C)$ and S are disjoint. Since $V(C)$ and S are disjoint, no vertex in C has a bump in the second position. It follows that no vertex $u \in V(C)$ has a bump in the third position either, since then the successor of u in C would have a bump in the second position. Therefore for each $u \in V(C)$ with $u = (u_1, \ldots, u_4)$, we have that $u_1 = 4$ or $u_4 = 4$. Since C has no edge uv with $u = (u_1, \ldots, u_4)$ and $v = (v_1, \ldots, v_4)$ where $u_4 = 4$ but $v_1 = 4$, it must be that either every vertex in C has 4 in the first position, forcing C to be the loop $\langle 4321 \rangle$, or every vertex in C has 4 in the fourth position, forcing C to be the loop $\langle 1234 \rangle$. It follows that $\tau(\mathrm{PSG}_4) \leq 10$.

To show $\tau(\mathrm{PSG}_4) \geq 10$, we construct a disjoint family $\mathcal{C}$ of cycles in PSG_4 with $|\mathcal{C}| = 10$. Note that the shift cycles partition $V(\mathrm{PSG}_4)$ into 6 subgraphs containing C_4. To produce four more cycles, we modify the partition as follows. As in Lemma 10, we split $\langle 1234, 2341, 3412, 4123 \rangle$ into the loop $\langle 1234 \rangle$ and the 3-cycle $\langle 2341, 3412, 4123 \rangle$. Similarly, we split $\langle 4321, 3214, 2143, 1432 \rangle$ into the loop $\langle 4321 \rangle$ and the 3-cycle $\langle 3214, 2143, 1432 \rangle$. Finally, we use the vertices in the shift cycles $\langle 1324, 3241, 2413, 4132 \rangle$ and $\langle 1423, 4231, 2314, 3142 \rangle$ to produce four 2-cycles: $\langle 1324, 3142 \rangle$, $\langle 3241, 2314 \rangle$, $\langle 2413, 4231 \rangle$, $\langle 4132, 1423 \rangle$; see Figure 2. We leave in place the other two shift cycles, giving

$$\begin{aligned}\mathcal{C} = \{\, & \langle 1234 \rangle, \langle 4321 \rangle, \langle 1324, 3142 \rangle, \langle 3241, 2314 \rangle, \langle 2413, 4231 \rangle, \langle 4132, 1423 \rangle, \\ & \langle 2341, 3412, 4123 \rangle, \langle 3214, 2143, 1432 \rangle, \langle 1243, 2431, 4312, 3124 \rangle, \langle 1342, 3421, 4213, 2134 \rangle \}.\end{aligned}$$

$\square$

Proposition 30. *We have* $\tau(\mathrm{PSG}_5) = 36$ *and* $a(\mathrm{PSG}_5) = 5! - \tau(\mathrm{PSG}_5) = 84$.

Proof. For the upper bound, let S be the set of vertices in PSG_5 consisting of the identity 12345, its reverse 54321, and vertices $(u_1, \ldots, u_5)$ with a bump at position 2. Let S_0 be the vertices $(u_1, \ldots, u_5)$ that have bumps in positions 2 and 4 and have $u_3 = 1$. Note that $|S| = 2 + (5!/3) = 42$ and $|S_0| = \binom{4}{2} = 6$ since a vertex $(u_1, \ldots, u_5) \in S_0$ is determined by the choice of the unordered pair $\{u_1, u_2\}$ from $\{2, 3, 4, 5\}$. Let $S' = S - S_0$ and note that $|S'| = |S| - |S_0| = 36$. We claim that S' is a cycle transversal. Let C be a cycle in PSG_5. Suppose that no vertex in C has a bump and let $u \in V(C)$ with $u = (u_1, \ldots, u_5)$. If $u = 54321$, then C and S' intersect. Otherwise $u_i < u_{i+1}$ for some i with $1 \leq i \leq 4$ and it follows

Condition	Threshold for k	Reference
$f(n,4,k) \geq 6$	13	Theorem 5 with $(r,t)=(4,3)$ and Theorem 2
$f(n,4,k) \geq \omega(1)$	15	Proposition 29 and Theorem 6
$f(n,4,k) \geq \Omega(n)$	$[15,19]$	Theorem 18
$f(n,4,k) = n$	$[15,23]$	Theorem 21
$f(n,5,k) \geq 6$	49	Theorem 5 with $(r,t)=(5,2)$ and Theorem 2
$f(n,5,k) \geq 8$	73	Theorem 5 with $(r,t)=(5,4)$ and Theorem 2
$f(n,5,k) \geq \omega(1)$	85	Proposition 30 and Theorem 6
$f(n,5,k) \geq \Omega(n)$	$[85,97]$	Theorem 18
$f(n,5,k) = n$	$[85,113]$	Theorem 21

Figure 3: Bounds on thresholds in 4-uniform and 5-uniform tournaments

that $(u_i,\ldots,u_r)$ is increasing since u has no bump. Moreover, since the successor v of u in C also has no bump, it follows that $(v_{i-1},\ldots,v_r)$ is increasing; eventually, we see that $12345 \in V(C)$ and so C and S' intersect.

Hence we may assume that some vertex in C has a bump, and by taking successors it follows that C contains a vertex u with a bump at position 2. If $u \notin S_0$, then C and S' intersect. So we may assume that $u \in S_0$ and so $u=(u_1,\ldots,u_5)$ where u has a bump at positions 2 and 4 with $u_3=1$. Note that $u_3<u_5$. Advancing twice along C gives a vertex v with $v=(v_1,\ldots,v_5)$ such that $v_1<v_3$ and v has a bump in position 2. It follows that $v \in S$. Also, since $v_3>v_1$ we cannot have $v_3=1$ and so $v \notin S_0$. It follows that $v \in S'$ and so C and S' intersect. Since C and S' intersect in all cases, S' is a cycle transversal and so $\tau(\mathrm{PSG}_5) \leq |S'| = 36$.

For the lower bound, we present a disjoint family $\mathcal{C}$ of cycles in PSG_r with $|\mathcal{C}|=36$ and $\bigcup_{C\in\mathcal{C}} V(C) = V(\mathrm{PSG}_5)$. The family was obtained via computer search. The family $\mathcal{C}$ has t_j cycles of length j, where $(t_1,\ldots,t_5)=(2,6,8,18,2)$. For readability, we group the cycles by their lengths.

$$
\begin{aligned}
\mathcal{C} = \{ & \langle 12345\rangle, \langle 54321\rangle, \\
& \langle 13254, 21435\rangle, \langle 14253, 31425\rangle, \langle 15243, 51423\rangle, \langle 34251, 32415\rangle, \langle 35241, 52413\rangle, \langle 45231, 53412\rangle, \\
& \langle 12435, 13245, 21354\rangle, \langle 13524, 25134, 51342\rangle, \langle 14523, 34125, 41352\rangle, \langle 15324, 53142, 41532\rangle \\
& \langle 32541, 25314, 52143\rangle, \langle 23514, 24135, 42351\rangle, \langle 24315, 43152, 42531\rangle, \langle 45312, 54231, 53421\rangle \\
& \langle 12354, 12534, 14235, 31245\rangle, \langle 12453, 13425, 23145, 21345\rangle, \langle 12543, 15423, 54123, 51243\rangle, \\
& \langle 32451, 34521, 34215, 32145\rangle, \langle 35421, 54312, 54132, 52431\rangle, \langle 43521, 45321, 54213, 53241\rangle, \\
& \langle 13452, 24513, 35124, 41235\rangle, \langle 23451, 34512, 45123, 51234\rangle, \langle 13542, 25413, 43125, 31254\rangle, \\
& \langle 23541, 35412, 53124, 41253\rangle, \langle 14352, 32514, 25143, 41325\rangle, \langle 24351, 42513, 35142, 51324\rangle, \\
& \langle 14532, 35214, 42135, 21453\rangle, \langle 24531, 45213, 52134, 31452\rangle, \langle 15342, 42315, 24153, 31524\rangle, \\
& \langle 25341, 52314, 34152, 41523\rangle, \langle 15432, 43215, 32154, 21543\rangle, \langle 25431, 53214, 42153, 31542\rangle, \\
& \langle 15234, 52341, 23415, 23154, 21534\rangle, \langle 43251, 43512, 45132, 51432, 14325\rangle \}
\end{aligned}
$$

□

It would be interesting to have general constructions of large disjoint families of cycles in PSG_r for general r. We note that disjoint cycles in PSG_r correspond to edge-disjoint cycles in the multigraph variant of PSG_{r-1} where each shift edge has multiplicity 2 (and other edges have multiplicity 1); like PSG_{r-1}, this multigraph variant is an Eulerian digraph and hence decomposes into cycles. Unfortunately, some cycles in the decomposition may be long, and the size of the decomposition may be small. We also note that each vertex in PSG_r is contained in an $(r-1)$-cycle. However, these shorter cycles do not seem to easily lead to disjoint families like the shift cycles do.

4 Density Results

The *falling factorial*, denoted $n_{(r)}$, is $n(n-1)\cdots(n-(r-1))$. In a *complete r-digraph*, every r-tuple of distinct vertices is an edge; equivalently, a complete r-digraph is an $(r, r!)$-tournament. The complete n-vertex r-digraph has $n_{(r)}$ edges.

In this section, we show that if G is an n-vertex r-digraph with $|E(G)| \geq (1-\frac{1}{r})n_{(r)}$, then G contains a path whose size increases with n. Conversely, for each positive ε, there are infinitely many r-digraphs G with $|E(G)| \geq (1-\frac{1}{r}-\varepsilon)n_{(r)}$, where $n = |V(G)|$, but all tight paths in G have size at most c, where c is a constant depending only on ε. In this sense, $1-\frac{1}{r}$ is the *density threshold* for growing tight paths in r-digraphs.

Lemma 31. *Let $0 < \varepsilon < 1$ and $r \geq 2$. For infinitely many n, there exists an n-vertex r-digraph G with $|E(G)| \geq (1-\frac{1}{r}-\varepsilon)n_{(r)}$ such that every tight path in G has at most $\frac{r^3}{\varepsilon}$ vertices.*

Proof. Fix a parameter t, and let n be a multiple of t. We construct an r-digraph G with vertex set $[n]$ as follows. Partition $[n]$ into t intervals $X_1, \ldots, X_t$ of equal size. Given an r-tuple $(u_1, \ldots, u_r)$, we put $(u_1, \ldots, u_r) \in E(G)$ if and only if $u_1 \in X_p$ and $u_j \in X_q$ for some j, p, q with $p < q$. By construction, if $v_1 \cdots v_s$ is a tight path in G, then we obtain a subsequence of vertices belong to parts with increasing indices of size at least $\lceil s/(r-1) \rceil$. It follows that $s \leq t(r-1)$.

To bound $|E(G)|$, consider choosing an r-set R from $[n]$ at random, and then selecting a random ordering of R to obtain an r-tuple $(u_1, \ldots, u_r)$. Let A be the event that the vertices in R belong to distinct parts, and note that $\Pr(A) \geq \prod_{j=0}^{r-1}\left(1-\frac{j}{t}\right) > \left(1-\frac{r}{t}\right)^r$. Let B be the event that $(u_1, \ldots, u_r) \in E(G)$. Conditioned on A, we have $\Pr(B|A) = 1-\frac{1}{r}$ since ordering R yields an edge in G unless the vertex belonging to the highest-indexed part is placed in the first position. It follows that $|E(G)| = \Pr(B) \cdot n_{(r)} \geq \Pr(B|A)\Pr(A) \cdot n_{(r)} > \left(1-\frac{1}{r}\right)\left(1-\frac{r}{t}\right)^r \cdot n_{(r)}$.

We choose t large enough so that $\left(1-\frac{1}{r}\right)\left(1-\frac{r}{t}\right)^r \geq 1-\frac{1}{r}-\varepsilon$; using calculus, we have that $t \geq \frac{r^2}{\varepsilon}$ suffices, so set $t = \lceil r^2/\varepsilon \rceil$. Since every tight path in G has at most $t(r-1)$ vertices and $t(r-1) \leq (\frac{r^2}{\varepsilon}+1)(r-1) \leq \frac{r^3}{\varepsilon}$, the claimed bound follows. ∎

By Lemma 16, if G has $\omega(n^{r-1})$ copies of $C_r^{(r)}$, then G has a path on $\omega(1)$ vertices. Let G be an n-vertex r-digraph. A set S of r vertices is *sparse* if $G[S]$ has fewer than $(1-\frac{1}{r})r!$ edges, is *dense* if $G[S]$ has more than $(1-\frac{1}{r})r!$ edges, and is *balanced* if $G[S]$ has exactly $(1-\frac{1}{r})r!$ edges. Since $G[S]$ contains a copy of $C_r^{(r)}$ when S is dense, n-vertex r-digraphs with $\omega(n^{r-1})$ dense r-sets contain growing paths.

Lemma 32. *For all positive integers r and s with $r \geq 2$, there is a constant $n_0 = n_0(r,s)$ such that if $n \geq n_0$ and G is an n-vertex r-digraph and $P_s^{(r)} \not\subseteq G$, then $|E(G)| < (1-\frac{1}{r}) \cdot n_{(r)}$.*

Proof. Let α, β, and γ be the fraction of r-sets in $V(G)$ which are sparse, balanced, and dense, respectively. Since each dense r-set contains a copy of $C_r^{(r)}$, it follows from Lemma 16 that the number of dense r-sets $\gamma\binom{n}{r}$ satisfies $\gamma\binom{n}{r} \leq (s-r+1)n_{(r-1)}$ or equivalently $\gamma \leq \frac{s-r+1}{n-r+1}r! = O(1/n)$. Hence $\gamma \to 0$ as $n \to \infty$.

Let H be the r-graph on $V(G)$ such that $e \in E(H)$ if and only if e is a balanced or dense r-set in G, let L be a maximum clique in H, and let $\ell = |L|$. Note that each r-set S of vertices in L is an edge in H and hence a dense or balanced r-set in G. It follows that $G[L]$ contains an ℓ-vertex (r,k)-tournament G_0, where $k = [(1-1/r)r!]$. Since $k > a(\mathrm{PSG}_r)$ and G_0 has no path on s vertices, it follows from Theorem 2 that ℓ is a bounded by a function of the constants r and s. Since H is an n-vertex r-graph with no clique of size $\ell+1$, it follows from de Caen's bound [3] that $|E(H)| \leq (1-1/\binom{\ell}{r-1}+o(1))\binom{n}{r}$. Since $|E(H)| = (\beta+\gamma)\binom{n}{r}$, the number of non-edges in H is $\alpha\binom{n}{r}$, and it follows that $\alpha\binom{n}{r} \geq (1/\binom{\ell}{r-1}-o(1))\binom{n}{r}$ and hence $\alpha/r! \geq (1/(r\ell_{(r-1)}))-o(1)$.

We compute

$$\begin{aligned}
|E(G)| &\le \alpha\binom{n}{r}\cdot\left(1-\frac{1}{r}-\frac{1}{r!}\right)r! + \beta\binom{n}{r}\cdot\left(1-\frac{1}{r}\right)r! + \gamma\binom{n}{r}\cdot r! \\
&= \left(\alpha\left(1-\frac{1}{r}-\frac{1}{r!}\right)+\beta\left(1-\frac{1}{r}\right)+\gamma\right)n_{(r)} \\
&= \left((1-\gamma)\left(1-\frac{1}{r}\right)+\gamma-\frac{\alpha}{r!}\right)n_{(r)} \\
&= \left(1-\frac{1}{r}-\frac{\alpha}{r!}+\frac{\gamma}{r}\right)n_{(r)} \\
&\le \left(1-\frac{1}{r}-\frac{1}{r\ell_{(r-1)}}+\frac{\gamma}{r}+o(1)\right)n_{(r)}.
\end{aligned}$$

Hence $|E(G)| < (1-(1/r))n_{(r)}$ provided that $\frac{\gamma}{r}+o(1) < \frac{1}{r\ell_{(r-1)}}$. Recalling that ℓ is a constant bounded by a function of r and s and $\gamma \to 0$ as $n \to \infty$, the lemma follows by taking n_0 large enough. $\square$

Taking the contrapositive of Lemma 32 gives the following.

Corollary 33. *For all positive integers r and s with $r \ge 2$, there is a constant $n_0 = n_0(r,s)$ such that if $n \ge n_0$ and G is an n-vertex r-digraph with $|E(G)| \ge (1-\frac{1}{r})n_{(r)}$, then $P_s^{(r)} \subseteq G$.*

Recall from Corollary 13 that when $k = (1-\frac{1}{r}-\frac{\varphi(r)-1}{r!})r!$, we have $k > a(\mathrm{PSG}_r)$ and so the (r,k)-tournaments have growing paths. These tournaments have $(1-\frac{1}{r}-\frac{\varphi(r)-1}{r!})n_{(r)}$ edges, corresponding to an edge density of $1-\frac{1}{r}-\frac{\phi(r)-1}{r!}$. For $r \ge 3$, we have $\phi(r) \ge 2$ and so at this density, growing paths are forced in tournaments but they are not quite yet forced in general r-digraphs.

5 Conclusions

Although obtaining the exact threshold on k for growing paths in (r,k)-tournaments appears to be difficult, obtaining the exact threshold on k for growing cycles in (r,k)-tournaments is easy.

Proposition 34. *Let $k = (1-\frac{1}{r}+\frac{1}{r!})r!$. If G is an n-vertex (r,k)-tournament, then $C_{n'}^{(r)} \subseteq G$ for some n' that grows with n.*

Proof. Let G be an (r,k)-tournament on vertex set $[n]$. Every r-set of vertices contains a copy of $C_r^{(r)}$. Label each r-set X by the canonical pattern of some edge in an r-cycle in $G[X]$. By Ramsey Theory, there is a subgraph H of G such that $|V(H)| = rn'$ where n' grows with n and every r-set in $V(H)$ is labeled with a common canonical pattern π. A tight cycle spanning H can be constructed as follows. Let $A_1,\ldots,A_r$ be a partition of $V(H)$ into r intervals of equal size such that $i<j$ implies $\max(A_i) < \min(A_j)$. There is a cyclic arrangement $v_1,\ldots,v_{rn'}$ of $V(H)$ such that each r-interval consists of vertices in distinct parts in $\{A_1,\ldots,A_r\}$, and for each i, we have that $\rho(v_{i+1},\ldots,v_{i+r})$ is a cyclic shift of π. Hence $\langle v_1,\ldots,v_{n'}\rangle$ is a spanning cycle in H whose size n' grows with n. $\square$

Since the $r!$ tuples on an r-set decompose into $(r-1)!$ copies of $C_r^{(r)}$, avoiding $C_r^{(r)}$ requires omitting at least $(r-1)!$ edges from each r-set. In fact, when $k = r!-(r-1)!$, it is possible to avoid all closed walks.

Proposition 35. *Let $k = (1-\frac{1}{r})r!$. For each n, there exists an n-vertex (r,k)-tournament with no closed walk.*

Proof. Construct an (r,k)-tournament G on $[n]$ by omitting $(u_1,\ldots,u_r)$ from $E(G)$ if and only if $u_1 = \max\{u_1,\ldots,u_r\}$. For each r-set, we omit $(r-1)!$ edges from G, and so G is an (r,k)-tournament with $k = r!-(r-1)! = (1-\frac{1}{r})r!$. Suppose W is a closed walk in G with $W = v_1 \ldots v_\ell$. Let $v_i = \max\{v_1,\ldots,v_\ell\}$ and note that v_i begins an edge in W, contrary to the definition of G. $\square$

Many interesting open problems remain; perhaps most compelling is the $(3,4)$-tournament conjecture (Conjecture 23). It would be notable progress to establish the $(3,4)$-tournament conjecture even for tournaments without closed walks.

Conjecture 36. *If G is a $(3,4)$-tournament with no closed walk, then G has a spanning path.*

Proving Conjecture 36 would imply $f(n, 3, 4) \geq \Omega(n^{1/3})$, improving Theorem 26. It would also be very interesting to generalize some of the techniques in Section 3.1 to larger r. Our next conjecture requests a generalization of Theorem 26.

Conjecture 37. *Let $k = (1 - \frac{1}{r})r!$. There is a positive ε such that $f(n, r, k) \geq \Omega(n^{\varepsilon})$.*

We note that Theorem 18 shows that when k is just one larger than the value in Conjecture 37, we have that $f(n, r, k)$ is linear in n.

6 AI Declaration

During the preparation of this work the authors used AI tools such as ChatGPT, Gemini, and Claude to assist in a search for large families of disjoint cycles in PSG_r and in an attempt to find a counter-example or proof of the $(3, 4)$-tournament conjecture. The text in this article was written by the authors and the authors take full responsibility for the content of this article.

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