Degree Ramsey numbers of graphs

William B. Kinnersley*, Kevin G. Milans†, Douglas B. West‡

Abstract

Let $H \overset{s}{\to} G$ mean that every s-coloring of $E(H)$ produces a monochromatic copy of G in some color class. Let the *s-color degree Ramsey number* of a graph G, written $R_\Delta(G;s)$, be $\min\{\Delta(H)\colon\ H \overset{s}{\to} G\}$. If T is a tree in which one vertex has degree at most k and all others have degree at most $\lceil k/2 \rceil$, then $R_\Delta(T;s) = s(k-1)+\epsilon$, where $\epsilon = 1$ when k is odd and $\epsilon = 0$ when k is even. For general trees, $R_\Delta(T;s) \le 2s(\Delta(T)-1)$.

To study sharpness of the upper bound, consider the *double-star* $S_{a,b}$, the tree whose two non-leaf vertices have have degrees a and b. If $a \le b$, then $R_\Delta(S_{a,b};2)$ is $2b-2$ when $a < b$ and b is even; it is $2b-1$ otherwise. If s is fixed and at least 3, then $R_\Delta(S_{b,b};s) = f(s)(b-1) - o(b)$, where $f(s) = 2s - 3.5 - O(s^{-1})$.

We prove several results about edge-colorings of bounded-degree graphs that are related to degree Ramsey numbers of paths. Finally, for cycles we show that $R_\Delta(C_{2k+1};s) \ge 2^s + 1$, that $R_\Delta(C_{2k};s) \ge 2s$, and that $R_\Delta(C_4;2) = 5$. For the latter we prove the stronger statement that every graph with maximum degree at most 4 has a 2-edge-coloring such that the subgraph in each color class has girth at least 5.

1 Introduction

Given a target graph G, classical graph Ramsey theory seeks a graph H such that every 2-edge-coloring of H produces a monochromatic copy of G. Such a graph H is a *Ramsey host* for G; we then write $H \to G$ and say that H *arrows* or *forces* G. More generally, we write $H \overset{s}{\to} G$ when every s-edge-coloring of $E(H)$ produces a monochromatic copy of G.

The classical *Ramsey number* of a graph G, written $R(G;s)$ in the general s-color setting, is the least n such that $K_n \overset{s}{\to} G$, guaranteed to exist by Ramsey's Theorem [24]. Note that

*Department of Mathematics, University of Illinois, Urbana, IL 61801, wkinner2@illinois.edu. Research supported by NSF grant DMS 08-38434, "EMSW21-MCTP: Research Experience for Graduate Students".

†Mathematics Dept., University of South Carolina, milans@math.sc.edu. Research supported by NSF grant DMS 08-38434, "EMSW21-MCTP: Research Experience for Graduate Students".

‡Department of Mathematics, University of Illinois, Urbana, IL 61801, west@math.uiuc.edu. Research supported by NSA grant H98230-10-1-0363.

$R(G;s) = \min\{|V(H)|:\ H \xrightarrow{s} G\}$. More generally, for any mononotone graph parameter ρ, the *ρ-Ramsey number* of G, written $R_\rho(G;s)$, is $\min\{\rho(H)\colon\ H \xrightarrow{s} G\}$. This notion has been studied with $\rho(G)$ being the number of edges $|E(G)|$, the clique number $\omega(G)$, the chromatic number $\chi(G)$, and the maximum degree $\Delta(G)$. Parameter Ramsey numbers are more difficult than ordinary Ramsey numbers in the sense that one may need to consider many more potential host graphs to determine whether $R_\rho(G) \le k$.

For the maximum size of a clique, Folkman [15] proved $R_\omega(G;2) = \omega(G)$. Nešetřil and Rödl [22] extended the result to s colors: always $R_\omega(G;s) = \omega(G)$.

Burr, Erdős, and Lovász [6] initiated the general study of parameter Ramsey numbers. They proved that the chromatic Ramsey number $R_\chi(G;s)$ always equals the Ramsey number of the family of homomorphic images of G. (For a family $\mathcal{G}$, the Ramsey number is the minimum number of vertices in a graph H such that every s-edge-coloring of H has a monochromatic copy of some graph in $\mathcal{G}$.) Since every homomorphic image of K_n contains K_n, it follows that $R_\chi(K_n;s) = R(K_n;s)$. They also conjectured that $\min\{R_\chi(G;s)\colon\ \chi(G) = k\}$ equals the easy lower bound $k^s + 1$, which was proved by Zhu [28, 29].

When $\rho(G) = |E(G)|$, we have the *size Ramsey number*, which we write as $\hat{R}(G)$ (for $s = 2$). Clearly, $\hat{R}(G) \le \binom{R(G)}{2}$ for all G. Erdős, Faudree, Rousseau, and Schelp [13] attributed to Chvatál the proof that equality holds when G is a complete graph. Beck [3] proved a conjecture of Erdős by showing that $\hat{R}(P_n) \le cn$, where c is a constant and P_n is the path on n vertices. Beck asked whether the size Ramsey number of graphs of bounded maximum degree grows linearly in the number of vertices. Chvatál, Rödl, Szemerédi, and Trotter [10] proved this behavior for the ordinary Ramsey number. The linear behavior holds for trees [16] and for cycles [11], but Rödl and Szemerédi [25] answered Beck's question in the negative by exhibiting an infinite family $\mathcal{F}$ of graphs with maximum degree 3 such that if G is an n-vertex graph in $\mathcal{F}$, then $\hat{R}(G) \ge c_1 n(\log n)^{c_2}$, where c_1 and c_2 are positive constants. (See [4, 11, 25] for further results on size Ramsey number.)

In this paper, we study the *degree Ramsey number*, $R_\Delta(G;s)$. Burr, Erdős, and Lovász [6] began its study (for $s = 2$). Their result on R_χ yields $R_\Delta(K_n;s) = R(K_n;s) - 1$. In Theorem 2.2 we extend the computation of $R_\Delta(K_{1,m};2)$ from [6] to the s-color setting, showing that $R_\Delta(K_{1,m};s)$ is $s(m-1)+1$ when m is odd and $s(m-1)$ when m is even ([6] also determined all graphs H such that $H \to K_{1,m}$). The computation for stars proves sharpness of an upper bound that holds for a large family of trees. In particular, $R_\Delta(T;s) \le R_\Delta(K_{1,m};s)$ whenever T is a tree with one vertex of degree at most m whose other vertices all have degree at most $\lceil m/2 \rceil$ (Theorem 2.4). Letting m be $2\Delta(T)$ gives the general upper bound $(T;s) \le 2s\Delta(T) - s + 1$, which can be improved to $2s(\Delta(T) - 1)$.

For general trees and large s, the upper bound is not far from sharp. To obtain a lower bound for trees having adjacent vertices of high degree, consider the *double-star* $S_{a,b}$, the tree having adjacent vertices of degrees a and b and no other non-leaf vertices. For fixed s least 3, we prove in Section 3 that $R_\Delta(S_{b,b}; s) = f(s)(b-1) - o(b)$, where $f(s) = 2s - 3.5 - O(s^{-1})$. The lower-bound argument colors any graph with smaller maximum degree probabilistically, so that with positive probability the resulting coloring has no monochromatic $S_{b,b}$. The situation is simpler when $s = 2$; we prove that $R_\Delta(S_{a,b}; 2)$ is $2b-2$ when $a < b$ and b is even, and is $2b-1$ otherwise when $a \le b$.

In Section 4, we study edge-colorings of bounded-degree graphs in relation to $R_\Delta(P_n; s)$. A short argument by Alon, Ding, Oporowski, and Vertigan [2] involving counting arguments and girth proves $R_\Delta(P_n; s) \le 2s$ for all n. They used a probabilistic construction to prove that equality holds when n is sufficiently large (for fixed s), showing the existence of an edge-coloring of any graph with maximum degree at most $2s-1$ that has no large monochromatic connected subgraph. With a more detailed look at the upper bound, we prove for fixed n that $H \xrightarrow{s} P_n$ for almost all graphs H with maximum degree at most $2s$. For the case $s = 2$, Thomassen [27] showed that the edges of any 3-regular graph can be 2-colored so that every monochromatic connected subgraph is contained in P_6. This yields $R_\Delta(P_n; 2) = 4$ for $n \ge 7$. Thomassen's proof was long; we give a short combinatorial proof of a weaker result implying that $R_\Delta(P_n; 2) = 4$ for $n > 15$. Although $R_\Delta(P_n; 2) = 3$ for $n \in \{4, 5\}$, it remains open whether $R_\Delta(P_6; 2)$ is 3 or 4. For short paths, an old result of Egawa et al. [12] yields $R_\Delta(P_4; s) \le 2s - 3$ for $s \ge 4$, and we show that always $R_\Delta(P_4; s) \ge s + 1$.

Section 5 concerns cycles. The values for C_3 follow from $R_\Delta(K_n; s) = R(K_n; s) - 1$, as noted earlier. Using Brooks' Theorem [5] and the fact that every 2^s-chromatic graph decomposes into s bipartite subgraphs, we obtain $R_\Delta(C_{2k+1}; s) \ge 2^s + 1$ for all odd cycles. For even cycles we obtain only $R_\Delta(C_{2k}; s) \ge 2s$ (Proposition 5.3); no better general lower bounds are known for even cycles.

In Theorem 5.5, we prove that $R_\Delta(C_4; 2) \ge 5$; equality holds from the result of Beineke and Schwenk [7] that $K_{5,5} \to C_4$. To obtain the lower bound, we prove the stronger statement that every graph with maximum degree at most 4 has a 2-edge-coloring such that the subgraph in each color class has girth at least 5. The best known general upper bounds for $s = 2$ are $R_\Delta(C_{2k}; 2) \le 96$ and $R_\Delta(C_{2k+1}; 2) \le 3458$ (see [18]). In addition to $R_\Delta(C_3; 2) = R_\Delta(C_4; 2) = 5$, the only other degree Ramsey number for cycles that is known exactly is $R_\Delta(C_3; 3) = 16$, which follows from $R(C_3; 3) = 17$.

The results of [18] show in addition that the 2-color degree Ramsey number of any graph with maximum degree 2 is bounded by 3458. It is natural to ask whether in general the 2-color (or s-color) degree Ramsey number is bounded by a function of the maximum degree.

2 Trees

We begin with the computation of $R_\Delta(K_{1,m}; s)$, applying classical results of graph theory. One upper bound uses a variation of a result of Bollobás, Saito, and Wormald [8]. They proved the existence of r-regular graphs without k-factors (for odd k); we will need such graphs with large girth. A *k-factor* of a graph G is a k-regular spanning subgraph of G. The *girth* of a graph G is the length of a shortest cycle in G. We use the result of Erdős and Sachs [14] that for each k and g there exist k-regular graphs with girth at least g.

Lemma 2.1. *If $r > k$ with k odd, and $g \geq 3$, then there exists a graph that is r-regular, has girth at least g, and has no k-factor.*

Proof. First consider even r. Let G be an r-regular graph with girth at least $g + 1$. If $|V(G)|$ is odd, then G has no k-factor. Otherwise, fix $v \in V(G)$. Since G is triangle-free, no neighbors of v are adjacent. Remove v and add a matching on its neighbors to create an r-regular graph G' with an odd number of vertices. A cycle C in G' that was not in G uses at least one new edge. If C uses only one new edge, then replacing it with two edges at v yields a cycle in G. If C uses at least two new edges, then C contains a path in G from one such edge to the next one, joining two neighbors of v, making C at least as long as a cycle in G. Hence every cycle in G' has length at least g.

Now consider odd r. We use a construction like that in [8] (for their case $\lambda = 1$). Let J be a graph in which all vertices have degree r except for one vertex x having degree $r - 1$. Construct a graph G by taking r copies of J and adding a new vertex y adjacent to all r copies of x. Suppose that G has a k-factor, H. Since r is odd and $r - 1$ is even, $|V(J)|$ is odd; thus J has no k-factor. Since a k-factor has degree k at *every* vertex, in H all r copies of J receive an edge from y, which contradicts $d_H(y) = k$. Thus G has no k-factor.

To complete the proof, it suffices to show that such a graph J exists with girth at least g (no cycles are added through y). Let F be an r-regular graph with girth at least $g + 1$, and fix $v \in V(F)$. Again the neighbors of v form an independent set. Form J by removing v and adding a matching of size $(r - 1)/2$ on the neighbors of v. By the argument in the first paragraph, J has girth at least g, and the vertex degrees are as desired. □

We will use Lemma 2.1 to determine $R_\Delta(K_{1,m}; s)$, but we do not yet need its full power; for the upper bound on $R_\Delta(K_{1,m}; s)$ we will not need large girth. The other results we need are *Vizing's Theorem* [26], which states that the edge-chromatic number of a graph G is at most $\Delta(G) + 1$, and *Petersen's Theorem* [23], which states that every regular graph of even degree decomposes into 2-factors.

Theorem 2.2. *If $s \geq 2$, then* $R_\Delta(K_{1,m}; s) = \begin{cases} s(m-1), & m \text{ even;} \\ s(m-1)+1, & m \text{ odd.} \end{cases}$

Proof. For the upper bound, the pigeonhole principle yields $K_{1,s(m-1)+1} \overset{s}{\to} K_{1,m}$. When m is even, we can improve the upper bound. By Lemma 2.1, there is an $(s(m-1))$-regular graph H having no $(m-1)$-factor; $H \overset{s}{\to} K_{1,m}$, since an s-edge-coloring of G with no monochromatic $K_{1,m}$ would be a decomposition of H into $(m-1)$-factors.

For the lower bound, let H be a graph with $\Delta(H) < s(m-1)$. By Vizing's Theorem, H is $s(m-1)$-edge-colorable, so $E(H)$ is the disjoint union of $s(m-1)$ matchings. Taking each color class to be the union of $m-1$ of these matchings yields an s-edge-coloring of H with no monochromatic $K_{1,m}$. When m is odd, we can improve the lower bound. For any graph H with maximum degree $s(m-1)$, let H' be an $s(m-1)$-regular supergraph of H. By Petersen's Theorem, H' decomposes into 2-factors. Taking each of s color classes to be the union of $(m-1)/2$ of these 2-factors yields an s-edge-coloring of H' with degree $m-1$ in each color at each vertex. □

Alon, Ding, Oporowski, and Vertigan [2] showed that $R_\Delta(P_n; s) = 2s$ for sufficiently large n. Thus Theorem 2.2 shows that it is "harder" to force stars than paths. Tao Jiang (unpublished) generalized the upper bound argument to show for any tree T that $R_\Delta(T) \leq 2s(\Delta(T)-1)$. For trees with only one vertex of large degree (including all those having exactly one vertex with degree exceeding 2), Jiang's argument can be improved. The upper bound meets the lower bound from stars and hence computes the exact value for these trees, since $G \subseteq G'$ implies $R_\Delta(G; s) \leq R_\Delta(G'; s)$. The lemma we need is a variation on a well-known fact.

Lemma 2.3. *Fix $r, q \in \mathbb{N}$ with $q \geq 2(r-1)$. If a graph H has average degree more than q, then H contains a subgraph with minimum degree at least r and average degree more than q.*

Proof. Let H be a smallest counterexample; let $n = |V(H)|$. If H has a vertex x with degree at most $r-1$, then $H - x$ has more than $\frac{1}{2}nq - (r-1)$ edges and hence has average degree more than q. Hence $H - x$ contains the desired subgraph. □

Theorem 2.4. *If T is a tree in which one vertex has degree at most k and all others have degree at most $\lceil k/2 \rceil$, then*

$$R_\Delta(T; s) \leq \begin{cases} s(k-1) & k \text{ even;} \\ s(k-1)+1 & k \text{ odd.} \end{cases}$$

Proof. Let $\epsilon = 1$ if k is odd and $\epsilon = 0$ if k is even. Let H be a regular graph having degree $s(k-1)+\epsilon$ and girth more than $|V(T)|$; by Lemma 2.1, we may also require H to have no $(k-1)$-factor when k is even. Given an s-edge-coloring of H, we seek a monochromatic subgraph H' that has a vertex x of degree at least k and has minimum degree at least r, where $r = \lceil k/2 \rceil$. In such a graph H', we can "grow" T from x by successively adding children. When we want to grow from a current leaf, it has $r-1$ neighbors in H' that (by the girth condition) are not already in the tree.

To obtain H', first consider odd k, so $r = (k+1)/2$. Since $\epsilon = 1$, in any s-edge-coloring of H some color class forms a spanning subgraph C with average degree more than $k-1$. Since $k-1 = 2(r-1)$, by Lemma 2.3 C has a subgraph H' with minimum degree at least r and average degree more than $k-1$. By the condition on average degree, H' has a vertex of degree at least k.

Now consider even k, with H as specified; note that $2(r-1) = k-2$. Since $\epsilon = 0$, some color class yields a spanning subgraph C with average degree at least $k-1$. Since H has no $(k-1)$-factor, C has a vertex of degree at least k. If it also has minimum degree at least $k/2$, then it is the desired monochromatic subgraph H'. Otherwise, delete a vertex x with degree in C at most $k/2-1$ (less than $(k-1)/2$). The average degree in $C-x$ is more than $k-1$. Now Lemma 2.3 yields a monochromatic subgraph H' with minimum degree at least r and average degree more than $k-1$. Again H' has a vertex of degree at least k. □

As noted previously, the bound in Theorem 2.4 holds with equality when T also satisfies $\Delta(T) = k$. We stated it for k as an upper bound on $\Delta(T)$ in order to obtain a general upper bound for trees. For any tree T, setting $k = 2\Delta(T) - 1$ in Theorem 2.4 yields $R_\Delta(T; s) \le 2s(\Delta(T) - 1) + 1$. Jiang's earlier unpublished observation by an argument like that of Theorem 2.4 improves this general bound by 1.

Theorem 2.5 (T. Jiang)**.** *If T is a tree, then $R_\Delta(T; s) \le 2s(\Delta(T) - 1)$.*

Proof. Let $r = \Delta(T)$, and let H be a $2s(r-1)$-regular graph with girth more than $|V(T)|$ (which exists by Erdős–Sachs [14]). Consider an s-edge-coloring of H. By the pigeonhole principle, some color class yields a monochromatic spanning subgraph C with average degree at least $2(r-1)$. By Lemma 2.3 (the proof is essentially the same when "more than" is changed to "at least" twice in Lemma 2.3), C has a monochromatic subgraph H' with minimum degree at least r. Now T can be grown inside H' from any vertex, as in the proof of Theorem 2.4. □

The result of Alon *et al.* [2] that $R_\Delta(P_n; s) = 2s$ shows that Theorem 2.5 is sharp when T is a path. For large s, in the next section we will use trees with exactly two non-leaf vertices to show that the coefficient $2s$ in Theorem 2.5 is almost sharp when s and $\Delta(T)$ are large.

3 Double-Stars

The *double-star* $S_{a,b}$ is the tree having two adjacent vertices of degrees a and b and no other non-leaf vertices. The double-star $S_{a,b}$ contains the star $K_{1,b}$. Surprisingly, when $s = 2$ and $a \le b$ always $R_\Delta(S_{a,b}; s) = R_\Delta(K_{1,b}; s)$, except when $a = b$ and b is even. This behavior fails for larger s, so we begin with the exact results for $s = 2$. Given graphs G and H, we say that an edge-coloring of H *avoids* G if no monochromatic copy of G appears in it.

Theorem 3.1. *If $a \le b$, then* $R_\Delta(S_{a,b}; 2) = \begin{cases} 2b-2 & \text{if } a < b \text{ and } b \text{ is even;} \\ 2b-1 & \text{otherwise.} \end{cases}$

Proof. Theorem 2.2 with $s = 2$ yields the lower bound except when $a = b$ and b is even. In that case, let H be any connected graph with $\Delta(H) \le 2b-2$, and let H' be a $(2b-2)$-regular connected graph that contains H. Starting with a vertex v, follow an Eulerian circuit C in H', coloring the edges by alternating red and blue along C. Each vertex of H' other than v receives an edge of each color with each passage through it, totaling $b-1$ edges of each color (this holds also for v if and only if $|E(H')|$ is even). Since every vertex other than v has degree at most $b-1$ in each color, it follows that $H' \not\to S_{b,b}$.

For the upper bound, we first prove $R_\Delta(S_{a,b}) \le 2b-1$ by showing that $H \to S_{b,b}$ whenever H is a triangle-free $(2b-1)$-regular graph. Consider a red/blue edge-coloring of H that avoids $S_{b,b}$. Call a vertex *red* when the majority (at least b) of its incident edges are red; otherwise it is *blue*. Without loss of generality, at least half the vertices are red. Since H is triangle-free, any red edge with red endpoints yields a red $S_{b,b}$, so each red edge has at least one blue endpoint. Since a red vertex lies on at least b red edges and a blue vertex lies on at most $b-1$ red edges, H has more blue vertices than red vertices, a contradiction.

We improve the upper bound by 1 when $a < b$ and b is even by showing that $H \to S_{a,b}$ for a particular $(2b-2)$-regular graph H. Form H using five disjoint vertex sets $S_0, \ldots, S_4$ of size $b-1$, making the neighborhood of each vertex in S_i consist of $S_{i-1} \cup S_{i+1}$ (indices taken modulo 5). (The graph H is often called the $(b-1)$-*blowup* of a 5-cycle.)

Consider a red/blue edge-coloring of H that avoids $S_{a,b}$. Each vertex is *red* (at least b incident red edges), *blue* (at least b incident blue edges), or *tied* ($b-1$ incident edges of each color). Not all are tied, since that would yield a regular subgraph with odd degree and odd order (since b is even). Without loss of generality, assume that S_0 contains a red vertex u.

Since H has no triangles and $a < b$, a red edge joining a red vertex to a red or tied vertex yields a red copy of $S_{a,b}$. Therefore, the neighbors of a red vertex along red edges are blue; similarly, neighbors of a blue vertex along blue edges are red. Since each S_i has size $b-1$, and colored vertices have at least b incident edges in their color, when S_i contains a colored vertex it follows that some vertex of S_{i+1} has the opposite color. Starting from our red vertex

u in S_0, we alternate finding blue and red vertices in successive sets to obtain a blue vertex v in S_0. Now in $S_1 \cup S_4$, the vertices adjacent to u along red edges are blue, and those adjacent to v along blue edges are red. This requires $|S_1 \cup S_4| \geq 2b$, but $|S_1 \cup S_4| = 2b - 2$. □

For $s > 2$, determining $R_\Delta(S_{a,b}; s)$ is more difficult. The proof of Theorem 3.1 does not extend, because it no longer need be that more than half of the vertices in the graph being colored have high degree in the same color. Nevertheless, Theorem 2.4 and Theorem 2.2 together compute $R_\Delta(S_{a,b}; s)$ when $b \geq 2a-1$; the value is $s(b-1)+1$ or $s(b-1)$, depending on the parity of b. Hence we focus our attention on $R_\Delta(S_{b,b}; s)$. The additional motivation for doing so is that this small tree shows that the general upper bound for trees in Theorem 2.5 is nearly sharp.

Definition 3.2. Given an s-edge-coloring of a graph H, we say that a vertex v is *major* in some color if it lies on at least b edges of that color and *minor* otherwise. A *minor edge* is an edge whose color is minor at both endpoints. Note that when the degree of a vertex exceeds $s(b-1)$, the vertex must be major in at least one color. Let $d^*(v)$ be the number of edges incident to v whose colors are minor at v.

Lemma 3.3. *Let H be a triangle-free graph. If an edge-coloring of H that has r minor edges avoids $S_{b,b}$, then*

$$|E(H)| + r = \sum_v d^*(v).$$

Proof. To avoid $S_{b,b}$, the color on each edge must be minor for at least one endpoint of the edge. Grouping the edges by the endpoints at which their colors are minor yields the sum on the right. Exactly r edges are counted twice. □

This lemma yields a slight improvement for $S_{b,b}$ of the general upper bound for trees.

Corollary 3.4. *If $s \geq 2$, then $R_\Delta(S_{b,b}; s) \leq 2(s-1)(b-1)+1$.*

Proof. Let H be a k-regular triangle-free graph, and consider an s-edge-coloring that avoids $S_{b,b}$. If $k > s(b-1)$, then each vertex is minor in at most $s-1$ colors. From Lemma 3.3, we then obtain $\frac{nk}{2} \leq \frac{nk}{2} + r \leq n(s-1)(b-1)$, which simplifies to $k \leq 2(s-1)(b-1)$. □

We next improve Corollary 3.4 asymptotically, for fixed s with $s \geq 3$. For clarity, we split the proof into several lemmas. The proof of the first closely mirrors that of a lemma by Alon [1], which he used to prove the existence of graphs having no "large" bipartite subgraphs. We bound the number of edges in a k-partite subgraph of a d-regular graph in terms of the sizes of the parts.

Definition 3.5. Let G be an n-vertex graph, and let $x_1, \ldots, x_k$ be nonnegative real numbers summing to 1. An $(x_1, \ldots, x_k)$-*subgraph* of G is a k-partite subgraph having partite sets of sizes $nx_1, \ldots, nx_k$.

A standard result in linear algebra states that the smallest eigenvalue of a real symmetric matrix A of order n is $\inf_{z \in \mathbb{R}^n} \frac{\langle Az, z \rangle}{\langle z, z \rangle}$, where $\langle x, y \rangle$ denotes the inner product of x and y.

Lemma 3.6. *Let G be a d-regular graph with n vertices and m edges, and let λ be its smallest eigenvalue. If $x_1, \ldots, x_k$ are nonnegative real numbers summing to 1, then no $(x_1, \ldots, x_k)$-subgraph of G has more than $(m - \lambda n/2) \sum_i x_i(1 - x_i)$ edges.*

Proof. Let A be the adjacency matrix of G. As remarked above, for every n-dimensional vector φ we have $\lambda \langle \varphi, \varphi \rangle \le \langle A\varphi, \varphi \rangle$. Consider any partition of $V(G)$ into $X_1, \ldots, X_k$, where $|X_i| = nx_i$, and let F be the $(x_1, \ldots, x_k)$-subgraph of G with partite sets $X_1, \ldots, X_k$ that includes all edges with endpoints in different parts. For $1 \le i \le k$, define the vector $\varphi^{(i)}$ by setting $\varphi^{(i)}_v = 1 - x_i$ when $v \in X_i$ and $\varphi^{(i)}_v = -x_i$ when $v \notin X_i$. Now

$$\begin{aligned} \left\langle A\varphi^{(i)}, \varphi^{(i)} \right\rangle &= 2 \sum_{uv \in E(G)} \varphi^{(i)}_u \varphi^{(i)}_v = d \sum_{v \in V(G)} (\varphi^{(i)}_v)^2 - \sum_{uv \in E(G)} \left(\varphi^{(i)}_u - \varphi^{(i)}_v\right)^2 \\ &= d \left\langle \varphi^{(i)}, \varphi^{(i)} \right\rangle - |[X_i, V(G) - X_i]|, \end{aligned}$$

where $[X, Y]$ denotes the set of edges joining X and Y. Summing over i now yields

$$\lambda \sum \left\langle \varphi^{(i)}, \varphi^{(i)} \right\rangle \le \sum \left\langle A\varphi^{(i)}, \varphi^{(i)} \right\rangle = d \sum \left\langle \varphi^{(i)}, \varphi^{(i)} \right\rangle - 2\,|E(F)|.$$

Thus

$$|E(F)| \le \frac{1}{2}(d - \lambda) \sum \left\langle \varphi^{(i)}, \varphi^{(i)} \right\rangle.$$

Since

$$\left\langle \varphi^{(i)}, \varphi^{(i)} \right\rangle = |X_i|\,(1 - x_i)^2 + (n - |X_i|)x_i^2 = nx_i(1 - x_i)^2 + n(1 - x_i)x_i^2 = nx_i(1 - x_i),$$

we have

$$|E(F)| \le \frac{1}{2}(d - \lambda)n \sum x_i(1 - x_i),$$

which simplifies to the claimed bound on $|E(F)|$. $\square$

To apply this lemma, we need regular graphs whose smallest eigenvalues are large. Lubotzky, Phillips, and Sarnak [19] defined a *Ramanujan graph* to be a regular graph whose smallest eigenvalue is at least $-2\sqrt{p-1}$, where p is the vertex degree. They constructed p-regular Ramanujan graphs for all primes p congruent to 1 modulo 4. Morgenstern [20] later constructed for each prime power p an infinite family of p-regular Ramanujan graphs G having girth at least $\frac{2}{3} \log_p |V(G)|$. Morgenstern's constructions and Lemma 3.6 together yield the following result.

Proposition 3.7. *Let p be a prime power, and let $x_1, \ldots, x_k$ be nonnegative real numbers summing to 1. For infinitely many n, there is a p-regular triangle-free n-vertex graph G having no $(x_1, \ldots, x_k)$-subgraph F with more than $\left(\sum_i x_i(1-x_i)\right)(m+n\sqrt{p-1})$ edges, where $m = np/2 = |E(G)|$.*

Lemma 3.3 and Proposition 3.7 together yield an asymptotic upper bound on $R_\Delta(S_{b,b}; s)$.

Lemma 3.8. *For fixed integer s, let U be the set of nonnegative s-tuples summing to 1. If $s \geq 3$, then $R_\Delta(S_{b,b}; s) \leq 2(M + o(1))(b-1)$, where*

$$M = \max_{y \in U} \frac{\sum_{i=1}^{s}(s-i)y_i}{2 - \sum_{i=1}^{s} y_i\left[1 - \frac{y_i}{\binom{s}{i}}\right]}.$$

Proof. Let H be a d-regular triangle-free n-vertex Ramanujan graph with m edges. Suppose that H has an s-edge-coloring avoiding $S_{b,b}$. We will show that $d \leq 2(M + o(1))(b-1)$. Taking $y_1 = 1$ and $y_2 = \cdots = y_s = 0$ yields $M \geq s/2$, so the bound already holds unless $d > s(b-1)$. Thus each vertex is major in at least one color.

Let C be the set of colors used. For $A \subseteq C$, let X_A be the set of vertices for which the set of major colors is A (note that $X_\varnothing = \varnothing$). For $1 \leq i \leq s$, let $Y_i = \bigcup\{X_A \colon\ |A| = i\}$. Let $y_i = |Y_i|/n$. Let F be the maximal (spanning) subgraph of H in which each X_A for $A \subseteq C$ is an independent set.

Let r be the number of minor edges in the given edge-coloring of H. Every edge joining two vertices that are major in exactly the same colors must be minor, since otherwise it would be the central edge of a monochromatic $S_{b,b}$. Thus $r \geq m - |E(F)|$. Also, each vertex in Y_i lies on at most $(s-i)(b-1)$ minor edges. With Lemma 3.3, we obtain

$$2m - |E(F)| \leq m + r = \sum_{v \in V(H)} d^*(v) \leq n(b-1)\sum_i (s-i)y_i. \tag{*}$$

To obtain an upper bound on m and hence on d, we need an upper bound on $|E(F)|$. When $y_1, \ldots, y_s$ are fixed, the upper bound in Proposition 3.7 is maximized when $|X_A|/n = y_{|A|}/\binom{s}{|A|}$ for each $A \subseteq C$. Thus, $|E(F)| \leq \sum_i y_i[1 - y_i/\binom{s}{i}](m + n\sqrt{d-1})$. Since $\sum y_i = 1$, we have $\sum_i y_i(1 - y_i/\binom{s}{i}) \leq 1$. Hence $|E(F)| \leq m(\sum_i y_i[1 - y_i/\binom{s}{i}]) + n\sqrt{d-1}$; this simplification will not change the asymptotics. Substituting into $(*)$ yields

$$m\left(2 - \sum_i y_i\left[1 - \frac{y_i}{\binom{s}{i}}\right]\right) - n\sqrt{d-1} \leq 2m - |E(F)| \leq n(b-1)\sum_i (s-i)y_i.$$

Since $m = nd/2$, this further simplifies to

$$d\left(2 - \sum_i y_i\left[1 - \frac{y_i}{\binom{s}{i}}\right] - \frac{2\sqrt{d-1}}{d}\right) \leq 2(b-1)\sum_i (s-i)y_i.$$

Thus

$$d \le 2(b-1)\frac{\sum_i (s-i)y_i}{2-\sum_i y_i[1-y_i/\binom{s}{i}]-o(1)},$$

where the $o(1)$ term tends to 0 as b tends to infinity (since $d > s(b-1)$). Since $\sum_i (s-i)y_i$ and $2-\sum_i y_i[1-y_i/\binom{s}{i}]$ are bounded, we may rewrite this as

$$d \le 2(b-1)\left(\frac{\sum_i (s-i)y_i}{2-\sum_i y_i[1-y_i/\binom{s}{i}]}+o(1)\right) \le 2(M+o(1))(b-1).$$

Finally, it suffices to show that there exists a d-regular Ramanujan graph when d is just a bit larger than this bound, which we call M'. Fix $\epsilon > 0$. For sufficiently large b, it follows from Proposition 3.7 and the Prime Number Theorem that there exist d-regular Ramanujan graphs with $M' < d < (1+\epsilon)M'$. □

We next show how to compute the value M in the statement of Lemma 3.8. The key insight is that the maximum is attained when all but at most two of the variables are zero.

Lemma 3.9. *For $s \ge 3$, with U being the set of nonnegative s-tuples summing to 1,*

$$M = \max_{y \in U} \frac{\sum_{i=1}^s (s-i)y_i}{2-\sum_{i=1}^s y_i\left[1-\frac{y_i}{\binom{s}{i}}\right]} = \frac{s-1}{2}\,\frac{s+\sqrt{s^2+s+2+4/(s-1)}}{s+1+2/s},$$

attained when $y_k = 0$ for $k \ge 3$ and $y_1 = 2-s+\sqrt{s^2-3s+6-8/(s+1)}$.

Proof. For notational convenience, let $f(y_1,\ldots,y_s) = \sum_i (s-i)y_i$ and $g(y_1,\ldots,y_s) = 2-\sum_i y_i\left(1-y_i/\binom{s}{i}\right)$. Suppose that the claim is false, and let $y_1,\ldots,y_s$ be real numbers maximizing f/g subject to $y \in U$.

We first claim that $y_j = 0$ when $j \ge 3$. If $y_j > 0$, then define $y'_1,\ldots,y'_s$ by $y'_j = 0$ and $y'_1 = y_1+y_j$, with $y'_i = y_i$ for $i \notin \{1,j\}$. Now $f(y'_1,\ldots,y'_s) = f(y_1,\ldots,y_s)+(j-1)y_j$ and

$$\begin{aligned} g(y'_1,\ldots,y'_s) &= g(y_1,\ldots,y_s)+y_1\left(1-\frac{y_1}{s}\right)+y_j\left(1-\frac{y_j}{\binom{s}{j}}\right)-(y_1+y_j)\left(1-\frac{y_1+y_j}{s}\right) \\ &= g(y_1,\ldots,y_s)-\frac{y_j^2}{\binom{s}{j}}+\frac{2y_1y_j+y_j^2}{s}. \end{aligned}$$

We claim that $f(y'_1,\ldots,y'_s)/g(y'_1,\ldots,y'_s) > f(y_1,\ldots,y_s)/g(y_1,\ldots,y_s)$. For positive real numbers a,b,c,d, the inequality $a/b < (a+c)/(b+d)$ holds if and only if $a/b < c/d$. Letting $a = f(y_1,\ldots,y_s)$, $b = g(y_1,\ldots,y_s)$, $c = (j-1)y_j$, and $d = \frac{2y_1y_j+y_j^2}{s}-\frac{y_j^2}{\binom{s}{j}}$, it suffices to show

$$\frac{f(y_1,\ldots,y_s)}{g(y_1,\ldots,y_s)} < \frac{(j-1)y_j}{\frac{2y_1y_j+y_j^2}{s}-\frac{y_j^2}{\binom{s}{j}}}.$$

We compute

$$\frac{(j-1)y_j}{\frac{2y_1y_j+y_j^2}{s}-\frac{y_j^2}{\binom{s}{j}}}=\frac{j-1}{\frac{2y_1+y_j}{s}-\frac{y_j}{\binom{s}{j}}}>\frac{s(j-1)}{2y_1+y_j}>\frac{s(j-1)}{2}\geq s,$$

but $f(y_1,\ldots,y_s)/g(y_1,\ldots,y_s)\leq s-1$, since the numerator is at most $s-1$ and the denominator is at least 1.

Thus $y_i=0$ for $i\geq 3$; consequently, $y_2=1-y_1$. It remains to choose $y\in[0,1]$ to maximize $\frac{f(y,1-y,0,\ldots,0)}{g(y,1-y,0,\ldots,0)}$, which simplifies to $s(s-1)h(y)$, where $h(y)=\frac{s-2+y}{s(s-1)+(s-1)y^2+2(1-y)^2}$. Note that $h(1)=1/(s+1)$ and $h(0)<1/(s+1)$. Setting $h'(y)=0$ yields a quadratic equation for y whose solution $\hat{y}$ is $2-s+\sqrt{s^2-3s+6-8/(s+1)}$. This value is 1 when $s=3$ (hence the requirement $s\geq 3$) and declines slowly toward $1/2$ as s increases, so it lies in $[0,1]$. Rationalizing the denominator in the expression for $h(\hat{y})$ yields $h(\hat{y})=\frac{1}{2}\frac{\sqrt{s^4-s^3+s^2+s-2}+s(s-1)}{s^3+s-2}$. Dividing the denominator by $s(s-1)$ and extracting a factor of $s-1$ from the numerator yields the claimed expression for M. □

Theorem 3.10. *For $s\geq 3$,*

$$R_\Delta(S_{b,b};s)\leq\left((s-1)\frac{s+\sqrt{s^2+s+2+4/(s-1)}}{s+1+2/s}+o(1)\right)(b-1),$$

where the $o(1)$ term tends to 0 as b tends to infinity.

Proof. Using the formula in Lemma 3.9 as the value of M, this becomes simply the statement of Lemma 3.8. □

Our lower bound for $R_\Delta(S_{b,b};s)$ asymptotically matches this upper bound. The value of $\hat{y}$ given in Lemma 3.8 will guide the construction.

Theorem 3.11. *For $s\geq 3$,*

$$R_\Delta(S_{b,b};s)\geq\left((s-1)\frac{s+\sqrt{s^2+s+2+4/(s-1)}}{s+1+2/s}-o(1)\right)(b-1),$$

where the $o(1)$ term tends to 0 as b tends to infinity.

Proof. Given a graph H with $\Delta(H)\leq d$, we construct a random s-coloring of $E(H)$ that avoids $S_{b,b}$ with positive probability when d is suitably chosen. We may assume that H is d-regular. Let C be a set of s colors.

We will assign each $v \in V(H)$ a set $c(v)$ of one or two colors in C, wanting only colors in $c(v)$ to be major for v. We try to match the bound from Lemma 3.9 by having the expected fractions of vertices that are major in one color or two colors be $\hat{y}$ and $1-\hat{y}$, respectively.

To this end, let $p = 2 - s + \sqrt{s^2 - 3s + 6 - 8/(s+1)}$. For each vertex v of H, let $\epsilon(v) = 1$ with probability p and $\epsilon(v) = 2$ otherwise. Next choose $c(v)$ uniformly at random from among all subsets of C with size $\epsilon(v)$. Given the resulting coloring of the vertices with color sets of size at most 2, we produce a coloring of $E(H)$, again probabilistically.

Fix $uv \in E(H)$. First consider $|c(u)| = |c(v)| = 1$. If $c(u) = c(v)$, then color uv with a random color from $C - c(u)$. If $c(u) \neq c(v)$, then give uv the color in $c(u)$ or $c(v)$, each with probability $1/2$. Next suppose $|c(u)| = 1$ but $|c(v)| = 2$. If $c(u) \subset c(v)$, then give uv the color in $c(v) - c(u)$. If instead $c(u) \cap c(v) = \varnothing$, then give uv the color in $c(u)$ with probability q and one of the colors in $c(v)$ with probability $(1-q)/2$ each, where q will be specified later. Finally, suppose $|c(u)| = |c(v)| = 2$. If $c(u) = c(v)$, then color uv randomly from $C - c(u)$. If $|c(u) \cap c(v)| = 1$, then give uv a color from the symmetric difference, each with probability $1/2$. Finally, if $c(u) \cap c(v) = \varnothing$, then color uv at random from $c(u) \cup c(v)$.

We claim that with positive probability every vertex v is major only in the colors in $c(v)$, when d and q are suitably chosen. It then holds by construction that no edge has a color that is major at both endpoints, so the coloring avoids $S_{b,b}$.

Fix a vertex v and a color c' not in $c(v)$. Let X be a random variable denoting the number of edges of color c' incident to v. Let the neighbors of v be $v_1, \ldots, v_d$. Now $X = Y_1 + \cdots + Y_d$, where Y_i is the indicator variable for the event that vv_i has color c'. If $|c(v)| = 1$, then $Y_i = 1$ can occur in four ways: $c(v_i) = c(v)$, $c(v_i) = \{c'\}$, $c(v) \subset c(v_i)$, and $|c(v_i)| = 2$ with $c(v) \not\subset c(v_i)$. Using conditional probability in each case,

$$\mathbb{P}\left[Y_i = 1 \,\middle|\, \epsilon(v) = 1\right] = \frac{1}{s-1} \cdot \frac{p}{s} + \frac{1}{2} \cdot \frac{p}{s} + 1 \cdot \frac{2(1-p)}{s(s-1)} + \frac{1-q}{2} \cdot (s-2)\frac{2(1-p)}{s(s-1)}.$$

Similarly, if $|c(v)| = 2$, then the cases in which vv_i can receive color c' are $c(v_i) = \{c'\}$, $c(v_i) = c(v)$, and $|c(v_i)| = 2$ with $|c(v_i) \cap c(v)|$ being 1 or 0. Thus

$$\mathbb{P}\left[Y_i = 1 \,\middle|\, \epsilon(v) = 2\right] = q \cdot \frac{p}{s} + \left(\frac{1}{s-2} + 1 + \frac{s-3}{4}\right) \cdot \frac{2(1-p)}{s(s-1)}.$$

Let p_1 and p_2 denote these two conditional probabilities. Since assigning color c' to vv_i is dangerous, we want to choose q to minimize $\max\{p_1, p_2\}$. As q increases, p_2 increases and p_1 decreases, so we choose q to produce $p_1 = p_2$, which requires

$$q = \frac{(s-2)(s+1) - 2s(1-p)}{2(s-2)(s-2+p)}.$$

We observed in Lemma 3.9 that $p > 1/2$, which easily implies $q > 0$, and comparing the numerator and denominator yields $q < 1$. Henceforth let $\hat{p}$ denote the common value of p_1

and p_2 when q is so chosen. Since $p_1 = p_2$, we now have $\mathbb{P}[Y_i = 1] = \hat{p}$. With $\hat{p} = \mathbb{P}[Y_i = 1]$ and H being d-regular, we have $\mathbb{E}[X] = \mathbb{E}[\sum_i Y_i] = d\hat{p}$.

Now let $d = \lfloor (1 - b^{-1/3})(b-1)/\hat{p} \rfloor$, so $\mathbb{E}[X] \le (1-b^{-1/3})(b-1)$. Since $b-1 \ge (1+\delta)\mathbb{E}[X]$, where $\delta = 1/(b^{1/3} - 1)$, the Chernoff Bound yields

$$\mathbb{P}[X > b-1] \le \mathbb{P}[X > (1+\delta)\mathbb{E}[X]] < e^{-\frac{\delta^2}{3}(1-b^{-1/3})(b-1)}.$$

Note that $\delta^2(1 - b^{-1/3})(b-1) = b^{1/3} + 1 + b^{-1/3}$. Let B_v be the event that v is major in a color outside $c(v)$. By the Union Bound,

$$\mathbb{P}[B_v] < (s-1)e^{-\frac{1}{3}(b^{1/3}+1+b^{-1/3})}.$$

The occurrence of B_v is determined by the color sets chosen at v and its neighbors and some choices made for edges incident to v. If v and w have a common neighbor, then B_v and B_w both make use of the color set chosen at that common neighbor. Nevertheless, B_v is mutually independent of the set of all B_u such that the distance between u and v is at least 3. Thus B_v is mutually independent of a set of all but at most d^2 other events. The symmetric version of the Lovász Local Lemma now states that $\mathbb{P}\left[\bigcap_v \overline{B}_v\right] > 0$ so long as

$$e \cdot (s-1)e^{-\frac{1}{3}(b^{1/3}+1+b^{-1/3})} \cdot (d^2 + 1) < 1.$$

Since d is bounded by a polynomial in b divided by a constant ($\hat{p}$ depends only on s), the inequality holds for sufficiently large b.

We have now shown that for some outcome of the vertex coloring (when b is sufficiently large), there is an outcome of the edge-coloring process that avoids $S_{b,b}$. It remains only to show that the degree for which we produced this coloring is $(2M - o(1))(b-1)$. Since $d = (1 - o(1))(b-1)/\hat{p}$, to complete the proof it suffices to prove that $\hat{p} = 1/(2M)$.

We show that $s(s-1)(\frac{1}{2M} - \hat{p}) = 0$. Since $\hat{y} = p$, we have $\frac{s(s-1)}{2M} = \frac{1}{2h(\hat{y})} = \frac{s(s-1)+(s-1)p^2+2(1-p)^2}{2(s-2+p)}$. Since $h'(\hat{y}) = 0$ yields

$$s(s-1) + (s-1)p^2 + 2(1-p)^2 = (s-2+p)[2p(s-1) - 4(1-p)], \tag{*}$$

we obtain $\frac{s(s-1)}{2M} = p(s-1) - 2(1-p)$. Using the formula for p_2, we have $s(s-1)\hat{p} = qp(s-1) + (\frac{1}{s-2} + 1 + \frac{s-3}{4})2(1-p)$. Now

$$s(s-1)\left(\frac{1}{2M} - \hat{p}\right) = (1-q)p(s-1) - \left(\frac{1}{s-2} + 2 + \frac{s-3}{4}\right)2(1-p).$$

Multiply by $2(s-2)(s-2+p)$, the denominator in the formula for q; this does not change whether the value on both sides is 0. For the right side, we compute

$$\begin{aligned}[2(s-2)(s-2+p) - (s-2)(s+1) + 2s(1-p)]p(s-1) - (s^2+3s-6)(1-p)(s-2+p)\\ = (s-2)[(s+1)p^2 + 2(s+1)(s-2)p - (s^2+3s-6)] \;=\; 0,\end{aligned}$$

since the last quadratic factor being 0 is equivalent to $(*)$. $\square$

Note that $p = 1$ when $s = 3$, so in this case the coloring for the lower bound makes every vertex major in exactly one color. For large s, the leading terms of the numerator and denominator in the expression we obtained for $R_\Delta(S_{b,b}; s)$ suggest that the value asymptotically equals the bound $2(s-1)(b-1)$ obtained in Corollary 3.4. However, the coefficient on $b-1$ is actually smaller; we need the two leading terms when multiplying by $s-1$.

Corollary 3.12. *For fixed s (with $s \geq 3$) and large b, we have $R_\Delta(S_{b,b}; s) = (c_s + o(1))(b-1)$, where $c_3 = 3$, $c_4 = 2 + \frac{2}{11}(1+\sqrt{210}) \approx 4.8166$, and in general $c_s = 2s - 3.5 + O(s^{-1})$.*

Proof. Divide the numerator and denominator of the expression for $2M$ by s. Then take the two leading terms of the series expansions for the square root and for the reciprocal of the denominator. This yields

$$\begin{aligned} 2M &= (s-1)\frac{1 + \sqrt{1 + s^{-1} + 2s^{-2} + 4s^{-2}(s-1)^{-1}}}{1 + s^{-1} + 2s^{-2}} \\ &= (s-1)(1 + 1 + \frac{1}{2}s^{-1} + O(s^{-2}))(1 - s^{-1} + O(s^{-2})) \ = \ 2s - 3.5 + O(s^{-1}). \end{aligned}$$

$\square$

4 Paths

For the path P_m, the degree Ramsey number $R_\Delta(P_m; s)$ is bounded, independent of m. Alon *et al.* [2] proved that $R_\Delta(P_m; s) = 2s$ when m is sufficiently large (for fixed s). The upper bound holds for all m; the argument is a special case of that given in Theorem 2.5. In fact, the main result of this section is that $H \xrightarrow{s} P_m$ for almost every graph H with maximum degree at most $2s$.

Given an edge-colored graph G, we call a maximal monochromatic connected subgraph a *slice* of G. For the lower bound, Alon *et al.* [2] proved that for each s, there is a constant c such that every graph with maximum degree less than $2s$ has an s-edge-coloring in which every slice has fewer than c edges. Thus $R_\Delta(P_m; s) \geq 2s$ for $m > c$.

Alon *et al.* did not give a sharp analysis of this value c. When $s = 2$, more precise results are known. Thomassen [27] showed that if $\Delta(H) \leq 3$, then there is a 2-edge-coloring of H in which every slice is a subgraph of P_6. Consequently, $R_\Delta(P_m; 2) = 4$ when $m \geq 7$. The other exact values are easy except for P_6.

Proposition 4.1. *$R_\Delta(P_m) = m - 1$ for $n \leq 3$, and $R_\Delta(P_m; 2) = 3$ for $m \in \{4, 5\}$.*

Proof. For P_2 use an edge, for P_3 an odd cycle. For $m \in \{4,5\}$, we have $R_\Delta(P_m) \ge 3$ because paths and cycles can be 2-edge-colored without three consecutive edges of the same color.

To show $R_\Delta(P_m) \le 3$ for $n \in \{4,5\}$, we prove that $H \to P_5$ when H is the Petersen graph. Consider a red/blue edge-coloring of H. Since H contains odd cycles, we may assume a red P_3. If there is no red P_4, then the four edges incident to the endpoints of this P_3 are all blue. Furthermore, the edges incident to the endpoints of these blue copies of P_3 must all be red, but this set of red edges contains P_4.

Hence we may assume a red P_4. If there is no red P_5, then the four edges incident to the endpoints of this P_4 are all blue, but these four edges form P_5. □

It remains open whether $R_\Delta(P_6; 2)$ is 3 or 4. It is easy to color the Petersen graph avoiding P_6; color a perfect matching blue, and the remaining edges form two disjoint red 5-cycles. Also the Heawood graph (the 3-regular 14-vertex $(3,6)$-cage, also known as the incidence graph of the Fano plane) has a 2-edge-coloring that avoids P_6, as in Fig. 1. (The graph in the bold color is $2P_5 + P_3$; the graph in the solid color is $2P_5 + K_{1,3}$.)

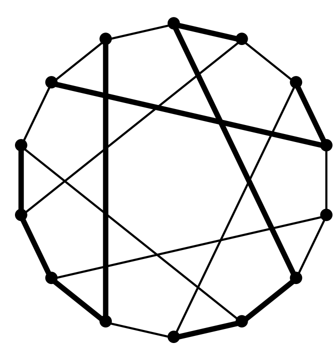


Figure 1: A 2-edge-coloring of the Heawood graph.

Conjecture 4.2. $R_\Delta(P_6; 2) = 4$.

Thomassen's proof that graphs with maximum degree 3 have 2-edge-colorings with every slice contained in P_6 is long. We give a short proof of the weaker result that there is a 2-edge-coloring in which every slice has at most 25 vertices, and more importantly every monochromatic path has at most 15 vertices; this yields $R_\Delta(P_m; 2) = 4$ for $m > 15$. In the bipartite case, it is easy to prove a result similar to Thomassen's.

Lemma 4.3. *Let P_7' be the tree obtained from P_7 by adding a leaf adjacent to the central vertex. If G is a bipartite graph with $\Delta(G) \le 3$, then G has a 2-edge-coloring in which all slices are subgraphs of P_7'.*

Proof. It suffices to prove the claim when G is 3-regular, in which case G has a perfect matching M. Let X and Y be the partite sets of G, and let $H = G - M$. Since H is 2-regular, it consists of disjoint even cycles.

First color $E(H)$. When the length of a component of H is an even multiple of 2, alternate two red edges and two blue edges so that the center of each monochromatic 3-vertex path is in X. When the length is an odd multiple of 2, do the same, except that one of the monochromatic paths has length 4. All leaves of all monochromatic paths in H lie in Y.

It remains to color M. A vertex $x \in X$ is an internal vertex of some monochromatic path in H; give the edge of M incident to x the opposite color. Monochromatic subgraphs of H grow only by adding edges of M at vertices of Y; a monochromatic copy of P_3 can only grow to a monochromatic copy of P_4 or P_5, and a monochromatic copy of P_5 can grow to as large as P_7' if it grows at each of its vertices in Y. □

Theorem 4.4. *If $\Delta(G) \le 3$, then G has a 2-edge-coloring in which each slice has at most 25 vertices and each monochromatic path has at most 15 vertices.*

Proof. We may assume that G is connected and that $G \ne K_4$, so Brooks' Theorem yields $\chi(G) \le 3$. By Lemma 4.3, we may assume that $\chi(G) = 3$.

Let $\{V_1, V_2, V_3\}$ be a proper 3-coloring of G with each vertex of V_3 having neighbors in both V_1 and V_2. For $u \in V_3$, choose neighbors $u_1 \in V_1$ and $u_2 \in V_2$. Let H be the graph on $V_1 \cup V_2$ obtained from $G - V_3$ by adding the edge u_1u_2 for each $u \in V_3$ such that $u_1u_2 \notin E(G)$. Since $\Delta(H) \le 3$, Lemma 4.3 yields a 2-edge-coloring f of H whose slices are subgraphs of P_7'.

Color $E(G)$ as follows. Edges joining V_1 and V_2 keep the color they have under f. Each remaining edge is incident to a vertex $u \in V_3$. Color uu_1 and uu_2 in G with the color that u_1u_2 has under f, and use the opposite color on a third edge incident to u, if it exists.

In progressing from H to G, monochromatic connected subgraphs can grow via edges incident to vertices in V_3. Two edges incident to $u \in V_3$ receive the same color only if their other endpoints form an edge of that color in H. Therefore, monochromatic connected subgraphs in H can grow only by adding leaves to monochromatic trees in H or by adding a common neighbor to vertices already adjacent in H. In particular, no two connected subgraphs of H having the same color can combine by adding edges of that color to V_3.

Because $\Delta(H) \le 3$ and each slice F of H is a tree with k vertices, where $k \le 8$, at most $3k - 2(k-1)$ edges can join F to vertices of V_3 in G in addition to the $k-1$ vertices of V_3 that can produced elements of $E(H) - E(G)$. Hence F grows by adding at most $2k + 1$ vertices to the original k vertices, so slices in the edge-coloring of G have at most 25 vertices.

Consider also monochromatic paths. Again they arise from connected subgraphs of H that are monochromatic under f. Such a path can have one edge at each end incident to V_3. Other edges incident to V_3 occur only by visiting $u \in V_3$ between u_1 and u_2. This may occur for each edge of a monochromatic path in H, since those edges need not exist in G. Since the maximum number of vertices in a monochromatic path under f is 7, we thus obtain 15 as the maximum number of vertices in a monochromatic path in the edge-coloring of G. □

For fixed s, we have $R_\Delta(P_m; s) = 2s$ when m is sufficiently large. In addition to finding the least m where equality holds, it would also be interesting to know the s-color degree Ramsey numbers for small paths. Since $P_3 = K_{1,2}$, Theorem 2.2 yields $R_\Delta(P_3; s) = s$, so the first nontrivial problem is for P_4. When H is bipartite, an s-edge-coloring that avoids P_4 is simply a decomposition of H into star-forests (forests in which each component is a star). The minimum number of star-forests needed to decompose H is the *star-arboricity* of H. The star-arboricity was determined for regular complete bipartite graphs by Egawa, Urabe, Fukuda, and Nagoya [12].

Lemma 4.5 ([12])**.** *When $s \geq 4$, the star arboricity is $s+1$ for both $K_{2s-3,2s-3}$ and $K_{2s-2,2s-2}$.*

Although [12] claims this result only for $s \geq 5$, in fact their elegant counting argument to show that $K_{2s-3,2s-3}$ does not decompose into s star-forests is valid also for $s = 4$.

Theorem 4.6. *If $s \geq 1$, then $R_\Delta(P_4; s) \geq s + 1$, with equality when $s \leq 4$. If $s \geq 4$, then $R_\Delta(P_4; s) \leq 2s - 3$.*

Proof. If $R_\Delta(P_4; s) \leq s$, then among the graphs with maximum degree s whose s-edge-colorings all have a monochromatic copy of P_4, choose H with fewest vertices. For $v \in V(H)$, there is an s-edge-coloring f of $H - v$ having no monochromatic copy of P_4. Since $\Delta(H) \leq s$, each neighbor w of v in H has degree at most $s - 1$ in $H - v$, and hence some color does not yet appear at w. Extend f to H by choosing such a color for the edge vw. All monochromatic subgraphs containing v are stars, and without v there is no monochromatic copy of P_4, so H in fact does not force P_4.

The general upper bound is immediate from Lemma 4.5 when $s \geq 4$. Since the star-arboricity of $K_{2s-3,2s-3}$ is $s+1$, every s-edge-coloring of $K_{2s-3,2s-3}$ fails to decompose it into star-forests and hence has a monochromatic P_4.

The upper bound $s + 1$ is trivial for $s = 1$, since $P_4 \subseteq K_{2,2}$. For $s = 2$, a 2-edge-coloring of $K_{3,3}$ has five edges in some color. A bipartite graph avoiding P_4 is a forest of stars, but the largest forest of stars in $K_{3,3}$ has four edges.

For $s = 3$, consider a 3-edge-coloring of $K_{4,4}$. There are 16 edges, so some color (say red) is used on at least six edges. A forest of stars with six edges on eight vertices has only two components and hence must be $2K_{1,3}$. Now the subgraph in the remaining two colors contains $K_{3,3}$, which forces P_4 as shown above. □

Lemma 4.5 states that $K_{2s-4,2s-4}$ has an s-edge-coloring with no monochromatic P_4 (the proof is an explicit construction of such a coloring for $K_{2s-2,2s-4}$). That is, the s-color bipartite Ramsey number of P_4 is $2s-3$. Although we proved only $s+1$ as a general lower bound, it may be that equality holds in $R_\Delta(P_4; s) \le 2s-3$ for $s \ge 4$.

Problem 4.7. Determine $R_\Delta(P_4; s)$.

Finally, we return to the s-color setting to show that $H \xrightarrow{s} P_m$ for almost every graph with maximum degree at most $2s$. Let $[n] = \{1, \ldots, n\}$.

Theorem 4.8. *Fix $m, s \in \mathbb{N}$. Let $\mathcal{G}_n$ be the family of graphs with vertex set $[n]$ and maximum degree at most $2s$. With $\mathcal{F}_n = \{H \in \mathcal{G}_n \colon H \stackrel{s}{\not\to} P_m\}$, we have $\lim_{n\to\infty} \frac{|\mathcal{F}_n|}{|\mathcal{G}_n|} = 0$.*

Proof. We first obtain a lower bound on $|\mathcal{G}_n|$. For each $2s$-tuple $(M_1, \ldots, M_{2s})$ of perfect matchings on $[n]$, form a graph H by letting $E(H) = \bigcup M_i$, and give each edge uv the color $\{j \colon uv \in M_j\}$. The number of ways to do this is z^{2s}, where z is the number of perfect matchings on $[n]$. Since $z = \prod_{j=1}^{n/2}(2j-1) = \frac{n!}{2^{n/2}(n/2)!}$, Stirling's Formula yields $z \sim \sqrt{2}\left(\frac{n}{e}\right)^{n/2}$. Dropping the $\sqrt{2}$, we observe that $z^{2s} > e^{sn\ln(n/e)}$ for sufficiently large n. The resulting structures consist of a graph in $\mathcal{G}_n$ with each edge colored by a subset of $[2s]$, and there are at most ns edges. Hence

$$|\mathcal{G}_n| > \frac{e^{sn\ln(n/e)}}{4^{s^2 n}} \ge e^{sn\ln(n/e)-(s^2\ln 4)n} \ge e^{sn\ln n - \alpha n},$$

where α is a constant.

Now we obtain an upper bound on $|\mathcal{F}_n|$. Let c be the maximum number of vertices in a graph with maximum degree at most $2s$ and diameter less than $m-1$. For each graph $H \in \mathcal{F}_n$, some s-edge-coloring of H witnesses $H \stackrel{s}{\not\to} P_m$. Every slice in this coloring has at most c vertices, since otherwise there is a monochromatic P_m. With H we associate this edge-coloring as a code; it is a decomposition of H into spanning subgraphs $(H_1, \ldots, H_s)$ such that $E(H_i)$ is the set of edges with color i. As we have noted, each component of each H_i has at most c vertices. Also, distinct graphs in $\mathcal{F}_n$ have distinct codes.

We bound $|\mathcal{F}_n|$ by bounding the number of codes that can be formed. Let $\mathcal{Q}$ be the set of all graphs having vertex set $[n]$, maximum degree at most $2s$, and components with at most c vertices. We have $|\mathcal{F}_n| \le |\mathcal{Q}|^s$.

We bound $|\mathcal{Q}|$ by building such a graph H' in three steps. First we specify a composition of n to record the numbers of vertices in the components of H'. That is, we specify positive integers $n_1, \ldots, n_k$ with sum n such that n_i is the number of vertices in the component of H' containing the least vertex of $[n]$ that is not in the earlier-indexed components. It is well known that there are 2^{n-1} compositions of n. We use only those whose parts are at most c, so there are at least n/c parts.

Next we record the distribution of vertices to components. For a graph with k components, we form a word of length $n-k$ from the characters in $[k]$. For i from 1 to k, there remain $N_i - (k-i+1)$ characters in the word, where $N_i = \sum_{j<i} n_i$. The vertices in the ith component are the least vertex not yet distributed plus the $n_i - 1$ vertices among the remainder whose relative position in the remaining word contains the character i. We only use words such that each character appears fewer than c times, but in any case there are at most $n^{n-n/c}$ such words, totalled over all choices of k. We write the bound as $e^{(1-1/c)n\ln n}$.

Finally, we record adjacencies within components. For each vertex v, we list its neighbors by recording a $2s$-tuple $(u_1, \ldots, u_{2s})$, where each u_j is a neighbor of v or is a dummy symbol 0. Many $2s$-tuples designate the same set of neighbors, but in any case there are at most c^{2s} possible $2s$-tuples for each vertex once the vertices have been distributed to components.

Multiplying all choices for the three steps, we have

$$|\mathcal{Q}| \le 2^{n-1} \cdot e^{(1-1/c)n\ln n} \cdot c^{2sn} \le e^{(n-1)\ln 2 + (1-1/c)n\ln n + 2sn\ln c}.$$

Now $|\mathcal{F}_n| \le e^{(1-1/c)sn\ln n + \beta n}$, where β is a constant. Thus $\frac{|\mathcal{F}_n|}{|\mathcal{G}_n|} \le e^{-(s/c)n\ln n + (\alpha+\beta)n} \to 0$. $\square$

5 Cycles

Since $C_3 = K_3$, the value of $R_\Delta(C_3; s)$ follows from the result of Burr, Erdős, and Lovász [6] that $R_\chi(G; s) = R(\mathrm{Hom}(G); s)$, where $\mathrm{Hom}(G)$ is the family of homomorphic images of G. If $H \xrightarrow{s} G$, then $\chi(H) \le \Delta(H) + 1$, so $R_\Delta(G; s) \ge R_\chi(G; s) - 1$. Since every homomorphic image of K_n contains K_n, it follows that $R_\chi(K_n; s) = R(K_n; s)$, and hence $R_\Delta(K_n; s) \ge R(K_n; s) - 1$. Since $K_{R(K_n;s)} \xrightarrow{s} K_n$, equality holds. In particular, $R_\Delta(C_3; 2) = 5$ and $R_\Delta(C_3; 3) = 16$.

It appears that $R_\Delta(C_n; s)$ behaves quite differently for odd and even n. The following Lemma is well known; Harary, Hsu, and Miller [17] noted the special case $k = 2$.

Lemma 5.1. *If $\chi(H) \le k^s$, then H decomposes into s graphs that are k-colorable.*

Proof. In a proper coloring f of H, encode the colors as k-ary s-tuples. Assign the edges of H to subgraphs $H_1, \ldots, H_s$ by putting uv in some H_i such that the colors of u and v differ in coordinate i. Each H_i is now properly colored by the values in the ith coordinate of f. $\square$

Proposition 5.2. *If $s \ge 2$ and $k \ge 1$, then $R_\Delta(C_{2k+1}; s) \ge 2^s + 1$.*

Proof. If $2 < \Delta(H) \le 2^s$ and H does not contain K_{2^s+1}, then H is 2^s-colorable (using Brooks' Theorem when $\Delta(H) = 2^s$). Hence H decomposes into s bipartite subgraphs, by Lemma 5.1. Therefore, $H \overset{s}{\not\to} C_{2k+1}$.

It remains to consider $H = K_{2^s+1}$. For $s = 2$, the edges of K_5 can be 2-colored to avoid any monochromatic fixed odd cycle. For $s > 2$, we proceed by induction on s. Partition the vertex set of K_{2^s+1} into two cliques, one of size $2^{s-1}+1$ and one of size 2^{s-1}. Color all edges between the cliques red. Use the other $s-1$ colors to color the edges within the cliques inductively while avoiding C_{2k+1}. $\square$

A result in bipartite Ramsey theory combines with Proposition 5.2 to show a qualitative difference between C_4 and odd cycles. Carnielli and Monte Carmelo [9] showed that $\lim_{s\to\infty} \frac{B(C_4;s)}{s^2} = 1$, where $B(G; s) = \min\{d\colon\ K_{d,d} \xrightarrow{s} G\}$. Thus $B(C_4; s)$ grows quadratically in s. Since $R_\Delta(G; s) \le B(G; s)$, it follows that $R_\Delta(C_4; s)$ grows at most quadratically in s. In contrast, Theorem 5.2 shows that the s-color degree Ramsey number of any odd cycle grows at least exponentially in s.

Lower bounds for even cycles are weaker. The technique involving chromatic numbers does not help, because large complete bipartite graphs are 2-chromatic but force long even cycles, by the bipartite Ramsey Theorem. The easiest way to avoid monochromatic long even cycles in a decomposition is to avoid all cycles. The *arboricity* of a graph G, denoted $\Upsilon(G)$, is the minimum number of forests in a decomposition of G.

Proposition 5.3. *If $k \ge 1$, then $R_\Delta(C_{2k}; s) \ge 2s$.*

Proof. Let G be a graph with $\Delta(G) \le 2s - 1$. By a famous result of Nash-Williams [21],

$$\Upsilon(G) = \max_{H\subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil \le \max_{H\subseteq G} \left\lceil \frac{\frac{1}{2}(2s-1)\,|V(H)|}{|V(H)|-1} \right\rceil \le s$$

so G decomposes into s forests. The resulting s-edge-coloring yields $R_\Delta(C_{2k}; s) \ge 2s$. $\square$

For $s = 2$, we have $R_\Delta(C_4; s) \ge 4$. Beineke and Schwenk [7] showed that $K_{5,5} \to C_4$, so $R_\Delta(C_4; 2) \le 5$. We will show that equality holds. It suffices to prove $R_\Delta(C_4; 2) > 4$. We will prove the stronger statement that for any graph G with $\Delta(G) \le 4$, some 2-edge-coloring of G avoids both C_3 and C_4. We will reach a contradiction by studying a smallest counterexample G. We show that such a graph G cannot contain various induced subgraphs, ultimately showing that G contains no triangle and no induced subgraph containing a 4-cycle.

Before proceeding, we need a lemma. We use $H \cdot e$ to denote the graph obtained from a graph H by contracting edge e.

Lemma 5.4 (Contraction Lemma)**.** *Let H be a graph with $\Delta(H) \le 4$. Let uv be an edge of H in no triangle, such that $d(u) + d(v) \le 6$. If $H \cdot uv$ has a 2-edge-coloring avoiding C_3 and C_4, then choosing either color for uv yields a 2-edge-coloring of H that avoids C_3 and C_4.*

Proof. We need only consider monochromatic triangles and 4-cycles through uv. There are none of the former, since uv lies on no triangles. With either color on uv, there are none of the latter, since they would contract to monochromatic triangles in $H \cdot uv$. $\square$

The subgraph of G induced by a vertex set S is denoted $G[S]$.

Theorem 5.5. $R_\Delta(C_4; 2) = 5$.

Proof. As mentioned earlier, it suffices to prove that every graph with maximum degree at most 4 has a 2-edge-coloring that avoids both C_3 and C_4. Let G be a smallest counterexample. We reach a contradiction by excluding various induced subgraphs from G.

In Figs. 2–6, the colors for the edges are solid and bold; dashed edges may have either of these colors. Vertices joined by dotted lines are nonadjacent. Every graph with fewer vertices than G has a solid/bold edge-coloring with no monochromatic C_3 or C_4; call this a *good coloring*. Often we will obtain a good coloring of G by "extending" a good coloring of an induced subgraph $G - S$ or a graph H that is not a subgraph of G; in the latter case, edges of H that are not in G are dropped. The proof of validity of the extension is always that every monochromatic cycle in the resulting edge-coloring of G is at least as long as some monochromatic cycle in the good coloring of the smaller graph, which is easily checked.

Step 1: G is K_4-free. Let S induce K_4. Extend a good coloring of $G - S$ by decomposing $G[S]$ into a solid P_4 and a bold P_4 and giving the edge leaving S at each vertex (if such an edge exists) the color of the P_4 having it as a leaf (see Fig. 2).

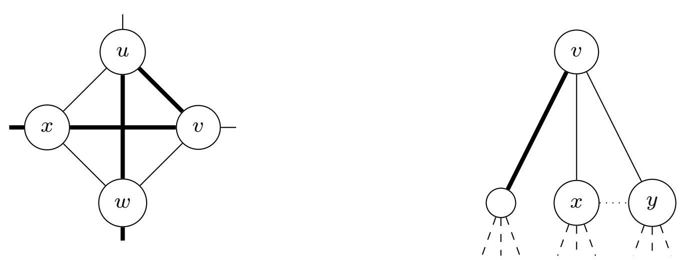


Figure 2: Left: G is K_4-free. Right: G is 4-regular.

Step 2: G is 4-regular. If $d(v) \le 2$, then a good coloring of $G - v$ extends by using each color on at most one edge at v. If $d(v) = 3$, then v has nonadjacent neighbors x and y, since G is K_4-free. Obtain H by adding xy to $G - v$. Extend a good coloring of H by giving vx and vy the color of xy in H and giving the third edge at v the opposite color (see Fig. 2).

Step 3: G is $K_{1,2,2}$-free. If G contains $K_{1,2,2}$ with vertex set S, then $G[S] = K_{1,2,2}$, since G is K_4-free. Extend a good coloring of $G - S$ by decomposing $G[S]$ into two copies of P_5 and giving to each edge leaving S the color of the P_5 ending there (see Fig. 3).

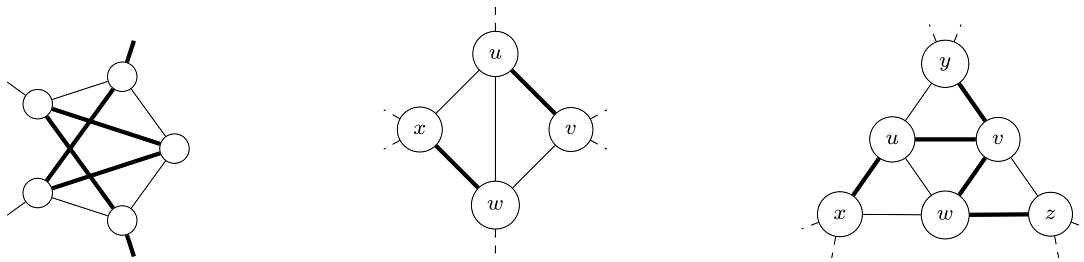


Figure 3: Left: G is $K_{1,2,2}$-free. Center, Right: G is K_4^--free.

Step 4: G is K_4^--free, where K_4^- is the graph obtained from K_4 by deleting one edge. Such a subgraph must be induced, since G is K_4-free; let S be a vertex set such that $G[S] = K_4^-$. Let x and v be the nonadjacent vertices in S, with $\{u, w\} = S - \{x, y\}$.

If u and v have no common neighbors outside S, and w and x have no common neighbors outside S (see Fig. 3 center), then let $H = (G \cdot uv) \cdot wx$, and let e be the edge that $G[S]$ contracts into. Extend a good coloring of H by giving the path through x, u, w, v the same color as e and giving the edges uv and wx the opposite color.

If neither reduction of this type is available, then we may assume by symmetry (and maximum degree 4) that u and v have a common neighbor y, and v and w have a common neighbor z (see Fig. 3 right). Let $S = \{u, v, w, x, y, z\}$. The graph $G[S]$ contains no additional edges, since G is $K_{1,2,2}$-free. Extend a good coloring of $G - \{u, v, w\}$ by using one color on $\{yu, uw, wx, vz\}$ and the opposite color on the rest. Note that each monochromatic path in $G[S]$ joining two vertices of $\{x, y, z\}$ has length at least 3.

Step 5: G is C_3-free. Suppose that $G[S] = C_3$, where $S = \{w, x, y\}$. Since G is K_4^--free, no two vertices of S have another common neighbor. Let u and v be the neighbors of w outside S. We consider two cases, depending on whether $uv \in E(G)$.

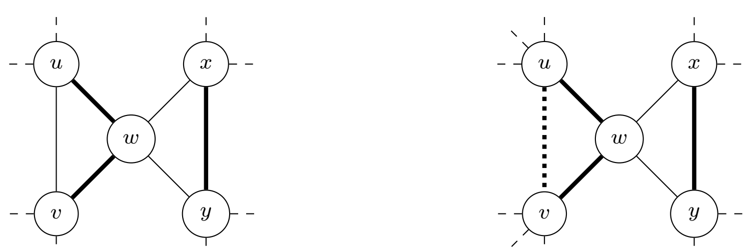


Figure 4: G is triangle-free.

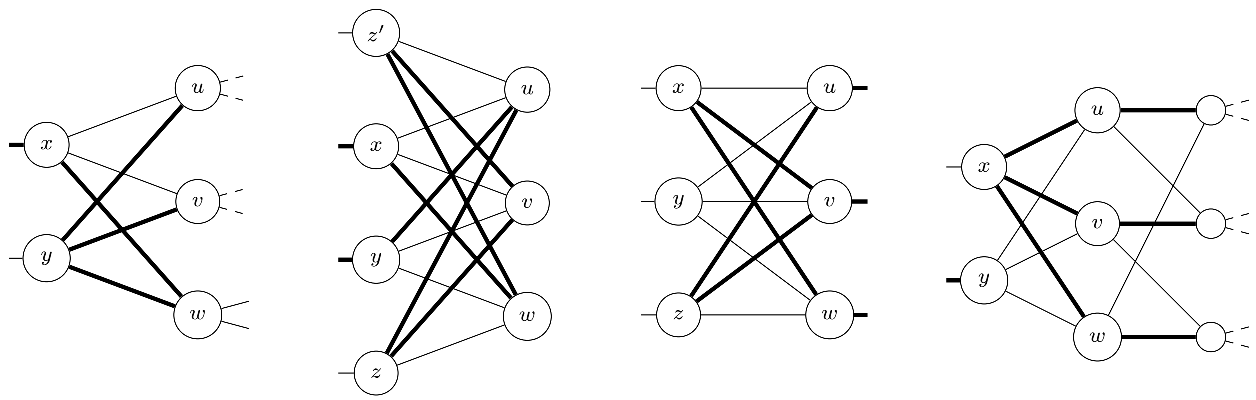
If $uv \in E(G)$, then let $H = (G - w) \cdot xy$. Extend a good coloring of H by giving xy the opposite color from uv (valid by the Contraction Lemma), then giving wx and wy the color of uv and wu and wv the color of xy (see Fig. 4 left). Monochromatic cycles through x, w, y

are long enough for the usual reason; monochromatic cycles through u, w, v are long enough because u and v have no other common neighbor, since G is K_4^--free.

If $uv \notin E(G)$, then let $H = (G + uv - w) \cdot xy$. Extend a good coloring of H by giving $\{uw, wv, xy\}$ the same color as uv and giving wx and wy the opposite color (see Fig. 4 right). Again the Contraction Lemma allows us to color xy arbitrarily, and for cycles through the other edges we have the usual reason.

Step 6: G is $K_{2,3}$-free. Suppose that $G[S] = K_{2,3}$; let X and Y be the partite sets of $G[S]$, with $X = \{x, y\}$ and $Y = \{u, v, w\}$. Since G is C_3-free, vertices of S having a common neighbor outside S must both lie in X or both in Y. We consider cases depending on whether two vertices of Y have a common neighbor outside S. Let xz and yz' be the edges leaving S from X; possibly $z = z'$.

If u and v have no common neighbor outside S, then let $H = (G - X) \cdot wb$, where a and b are the neighbors of w outside S. Extend a good coloring of H by giving $\{ux, vx, yz', wb\}$ the same color as wa and giving the opposite color to the remaining edges incident to X (see Fig. 5a). The Contraction Lemma allows us to color wb as specified, and then all monochromatic cycles containing an edge incident to X must pass through u and v; these are long enough because u and v have no common neighbor outside S. The possibility of $z = z'$ is irrelevant in this case.



(a) (b) (c) (d)

Figure 5: G is $K_{2,3}$-free.

Therefore, we may assume that every two vertices in Y have a common neighbor. Possibly some vertex outside S is adjacent to all of Y. If there are two such vertices z and z', then let $S' = S \cup \{z, z'\}$; now $G[S'] = K_{4,3}$. In this case, extend a good coloring of $G - S'$ by decomposing $G[S']$ into two copies of P_7 and coloring the edges leaving S' with the color of the P_7 whose endpoint they are adjacent to (see Fig. 5b). It does not matter whether the vertices in $S' - Y$ have common neighbors outside S'.

If only one vertex outside S is adjacent to all of Y, call it z and let $S' = S \cup \{z\}$; now $G[S'] = K_{3,3}$ (see Fig. 5c). Each vertex of S' has one neighbor outside S'. Since G is now $K_{4,3}$-free, we may assume that no pair in Y except possibly $\{u, w\}$ and no pair in $\{x, y, z\}$ except possibly $\{x, z\}$ has a common neighbor outside S'. Now extend a good coloring of $G - S'$ by making the edges leaving S' from Y and the path through u, z, v, x, w bold and making the edges leaving S' from $\{x, y, z\}$ and the remaining edges of $G[S']$ solid.

The final possibility is that each pair in Y has a common neighbor outside S, but no vertex outside S is adjacent to all of Y. Extend a good coloring of $G - S$ by decomposing the set of edges incident to S into two copies of $K_2 + T$, where T is the 7-vertex tree obtained by subdividing each edge of $K_{1,3}$ (see Fig. 5d).

Step 7: G does not contain C_4. Since G is K_4-free and K_4^--free, any 4-cycle is an induced subgraph. If $G[S] = C_4$ for some S, let the vertices be u, v, w, x in cyclic order. Since G is C_3-free and $K_{2,3}$-free, no two vertices of S have a common neighbor outside S. Let v' and v'' be the neighbors of v outside S, and let w' and w'' be the neighbors of w outside S.

Obtain H from $G - \{v, w\}$ by adding the edges $v'v''$ and $w'w''$ and contracting the edge ux. Note that $v'v'', w'w'' \notin E(G)$, since G is C_3-free. By the Contraction Lemma, when extending a good coloring of H we may give ux whichever color we want. We give vv' and vv'' the color of $v'v''$, and we give ww' and ww'' the color of $w'w''$.

For the remaining edges, we consider two cases. If $v'v''$ and $w'w''$ have the same color in H, then give that color to ux and give the opposite color to the rest of $G[S]$ (see Fig. 6 left). If $v'v''$ and $w'w''$ have opposite color in H, then give the color of $v'v''$ to vw and xw, and give the color of $w'w''$ to xu and vu (see Fig. 6 right). By the usual arguments, the resulting monochromatic cycles are long enough, using the fact that no two vertices of S have a common neighbor outside S.

Figure 6: G is C_4-free.

Since G has neither a C_3 nor a C_4, every 2-edge-coloring avoids both. $\square$

References

[1] N. Alon, Bipartite subgraphs, *Combinatorica* 16 (1996), no. 3, 301–311.

[2] N. Alon, G. Ding, B. Oporowski, and D. Vertigan, Partitioning into graphs with only small components, *J. Combin. Theory Ser. B* 87 (2003), no. 2, 231–243.

[3] J. Beck, On size Ramsey number of paths, trees and cycles I, *J. Graph Theory* 7 (1983), 115–130.

[4] J. Beck, On size Ramsey number of paths, trees and cycles II. *Mathematics of Ramsey Theory* (Springer), *Algorithms and Combin.* 5 (1990), 34–45.

[5] R. L. Brooks, On colouring the nodes of a network, *Proc. Cambridge Philosophical Society, Math. Phys. Sci.* 37 (1941), 194197.

[6] S. Burr, P. Erdős, and L. Lovász, On graphs of Ramsey type, *Ars Combinatoria* 1 (1976), 167–190.

[7] L. W. Beineke and A. J. Schwenk, On a bipartite form of the Ramsey problem, *Congressus Numerantium* 15 (1975), 17–22.

[8] B. Bollobás, A. Saito, and N.C. Wormald, Regular factors of regular graphs, *J. Graph Theory* 9 (1985), 97–103.

[9] W. A. Carnielli and E. L. Monte Carmello, On the Ramsey problem for multicolor bipartite graphs, *Advances in Applied Mathematics* 22 (1999), no. 1, 48–59.

[10] V. Chvátal, V. Rödl, E. Szemerédi, and W. T. Trotter, The Ramsey number of a graph with bounded maximum degree. *J. Combin. Theory Ser. B* 34 (1983), no. 3, 239–243.

[11] J. Donadelli, P.E. Haxell, and Y. Kohayakawa, A note on the size-Ramsey number of long subdivisions of graphs, *RAIRO-Inf. Theor. Appl.* 39 (2005), 191–206.

[12] Y. Egawa, M. Urabe, T. Fukuda, and S. Nagoya, A decomposition of complete bipartite graphs into edge-disjoint subgraphs with star components, *Discrete Math.* 58 (1986), 93–95.

[13] P. Erdős, R.J. Faudree, C.C. Rousseau, and R.H. Schelp, The size Ramsey number, *Period. Math. Hungar.* 9 (1978), 145–161.

[14] P. Erdős and H. Sachs, Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl. (German) *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* 12 (1963), 251–257.

[15] J. Folkman, Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM J. Appl. Math.* 18 (1970), 19–24.

[16] J. Friedman and N. Pippenger, Expanding graphs contain all small trees. *Combinatorica* 7 (1987), no. 1, 71–76.

[17] F. Harary, D. Hsu, and Z. Miller, The biparticity of a graph, *J. Graph Theory* 1 (1977), no. 2, 131–133.

[18] T. Jiang, K.G. Milans, and D.B. West, Degree Ramsey number of cycles is bounded, in preparation.

[19] A. Lubotzky, R. Phillips, and P. Sarnak, Explicit expanders, and the Ramanujan conectures, *Proc. 18th ACM STOC* (1986), 240–246.

[20] M. Morgenstern, Existence and explicit constructions of $(q + 1)$-regular Ramanujan graphs for every prime power q, *J. Combinatorial Theory Ser. B* 62 (1994), 44–62.

[21] C. St. J. A. Nash-Williams, Decomposition of a finite graph into forests, *J. London Math. Soc.* 39 (1964), 12.

[22] J. Nešetřil and V. Rödl, The Ramsey property for graphs with forbidden complete subgraphs. *J. Combinatorial Theory Ser. B* 20 (1976), no. 3, 243–249.

[23] J. Petersen, Die Theorie der regulären Graphen, *Acta Math.* 15 (1891), 193–220.

[24] F.P. Ramsey, On a problem of formal logic, *Proc. Lond. Math. Soc.* 30 (1930), 264–286.

[25] V. Rödl and E. Szemerédi, On size Ramsey numbers of graphs with bounded maximum degree, *Combinatorica* 20 (2000), 257–262.

[26] V.G. Vizing, On an estimate of the chromatic class of a p-graph, *Diskret. Anal.* 3 (1964), pp. 25–30.

[27] C. Thomassen, Two-coloring the edges of a cubic graph such that each monochromatic component is a path of length at most 5, *J. Combin. Theory Ser. B* 75 (1999), 100–109.

[28] X. Zhu, Chromatic Ramsey numbers, *Discrete Math.* 190 (1998), 215–222.

[29] X. Zhu, Fractional Hedetniemi's conjecture is correct, preprint.