Online coloring blowups of a known graph

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West Virginia University

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A motivating problem

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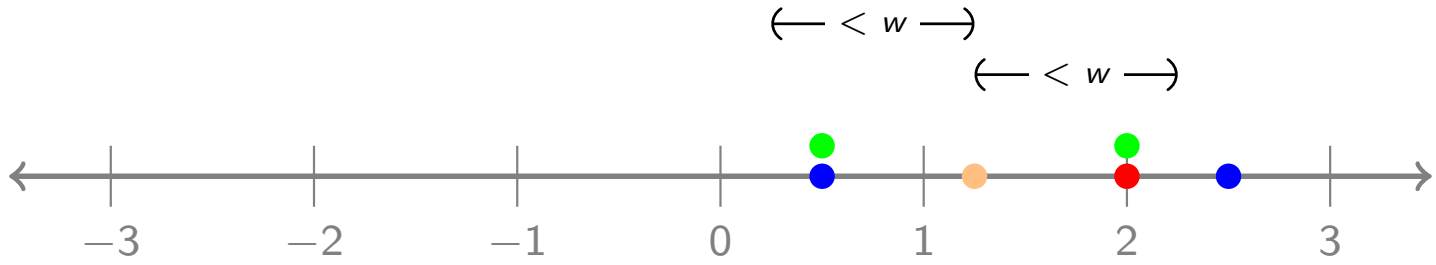

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- The greedy algorithm uses at most $2w - 1$ colors.

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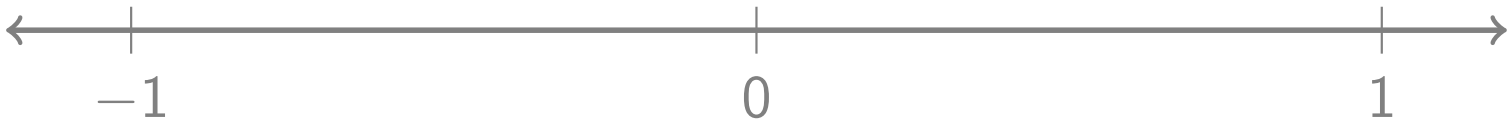

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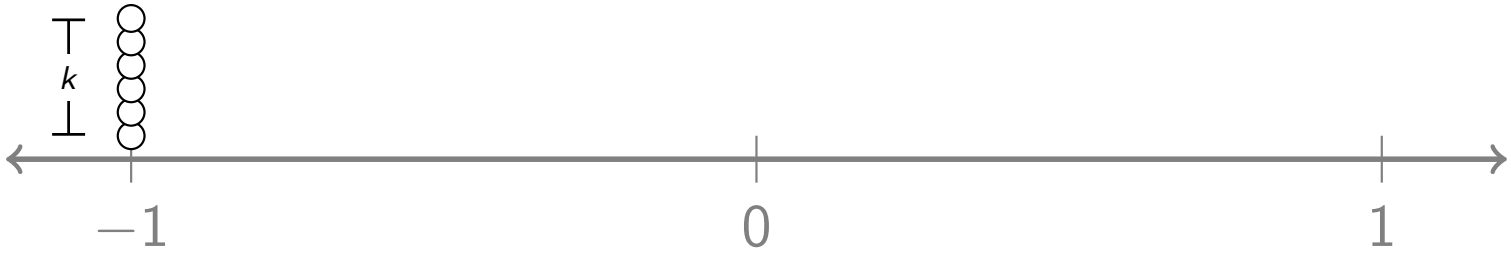

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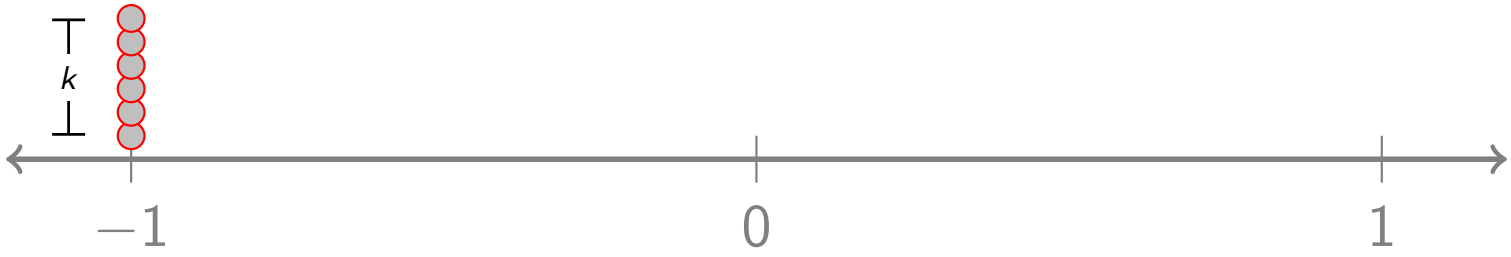

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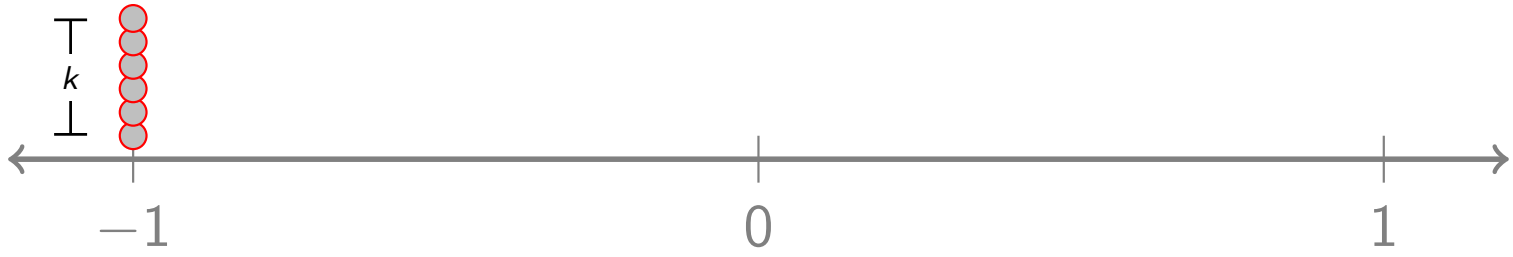

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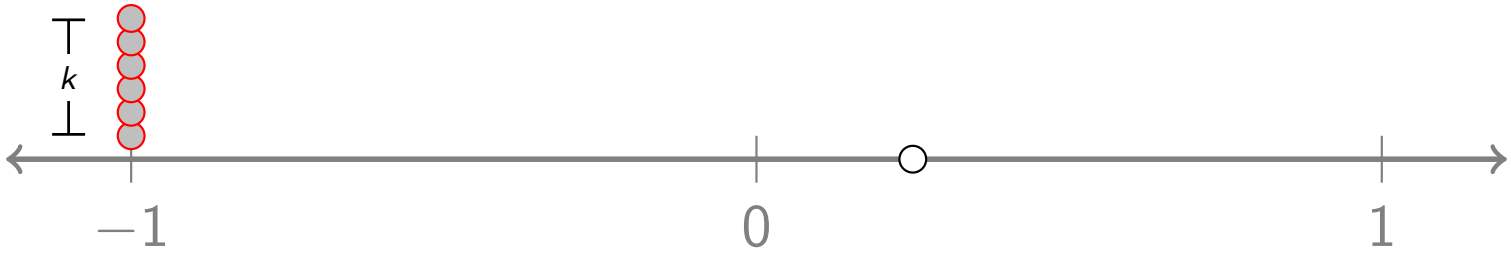

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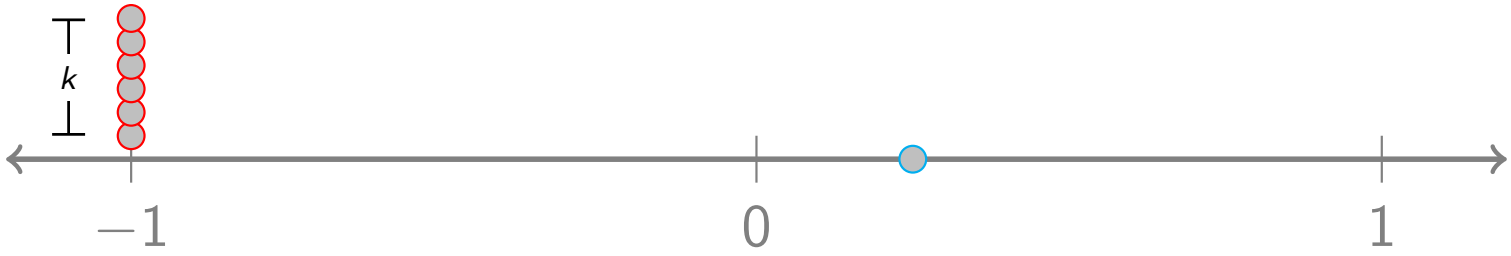

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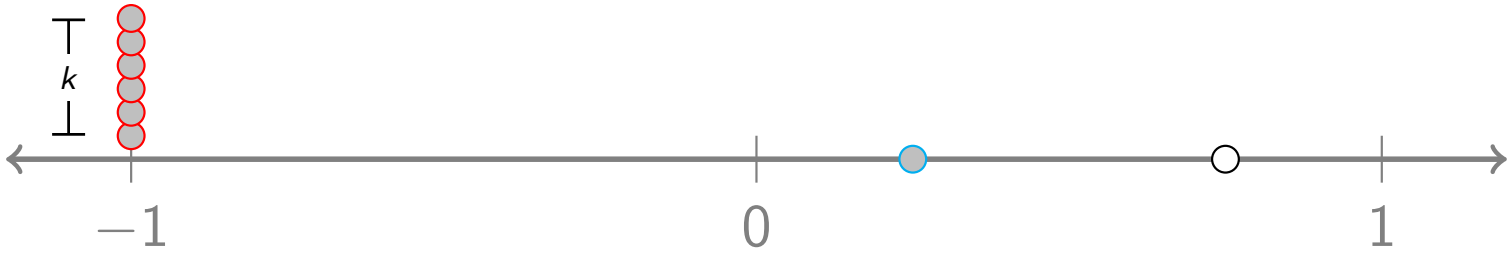

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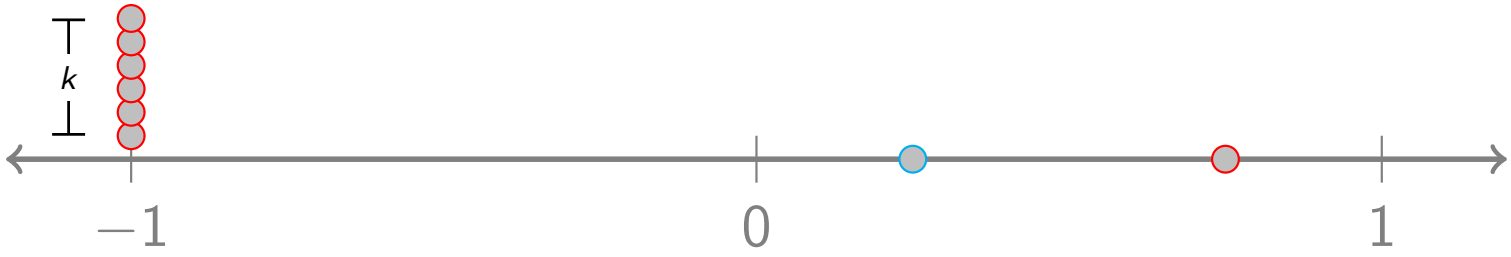

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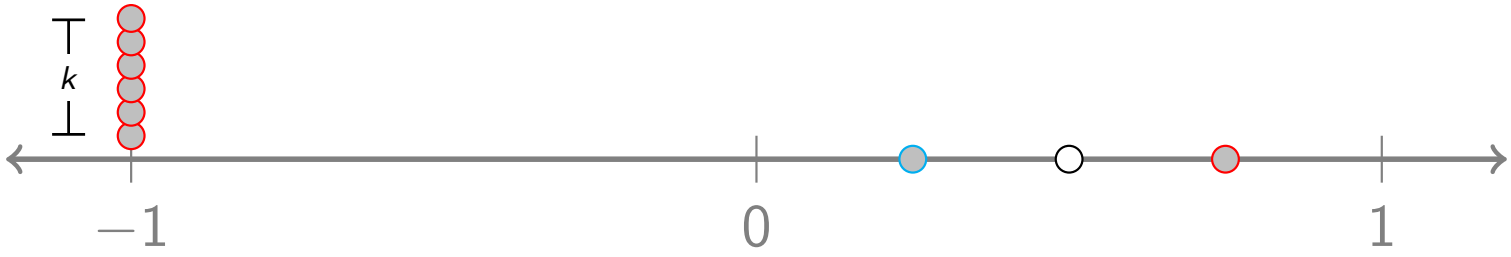

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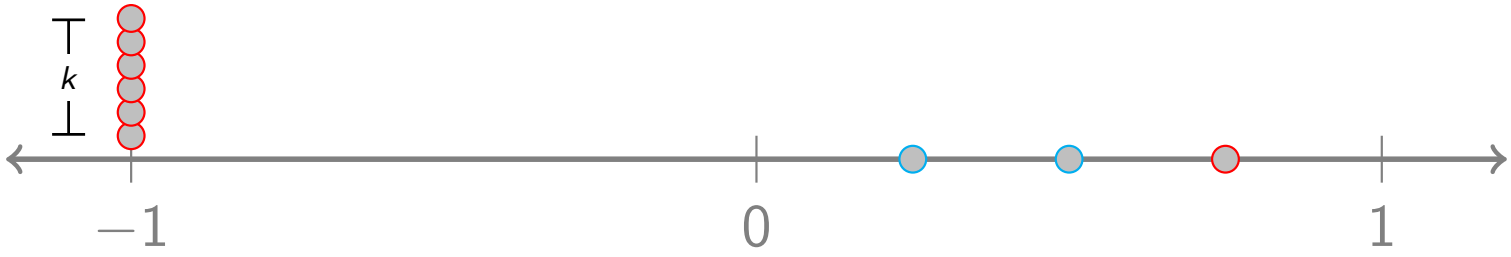

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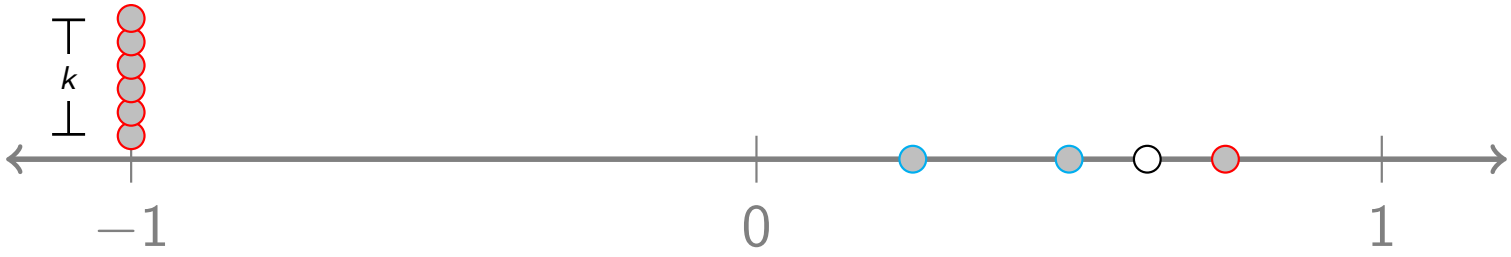

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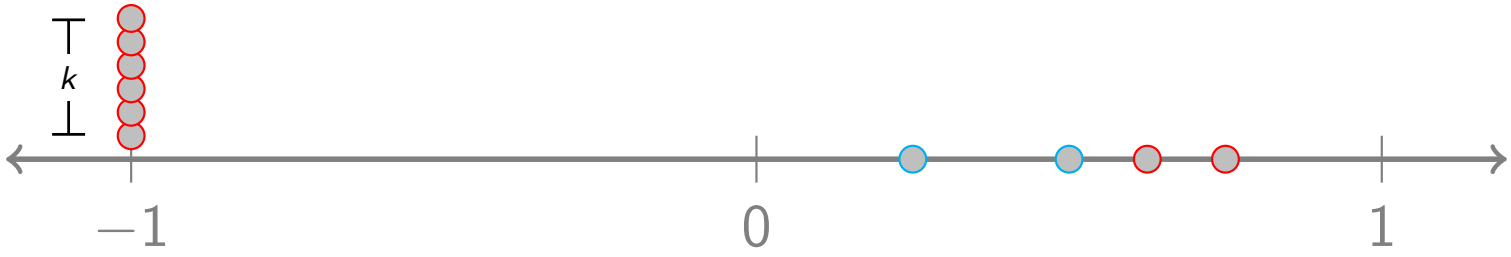

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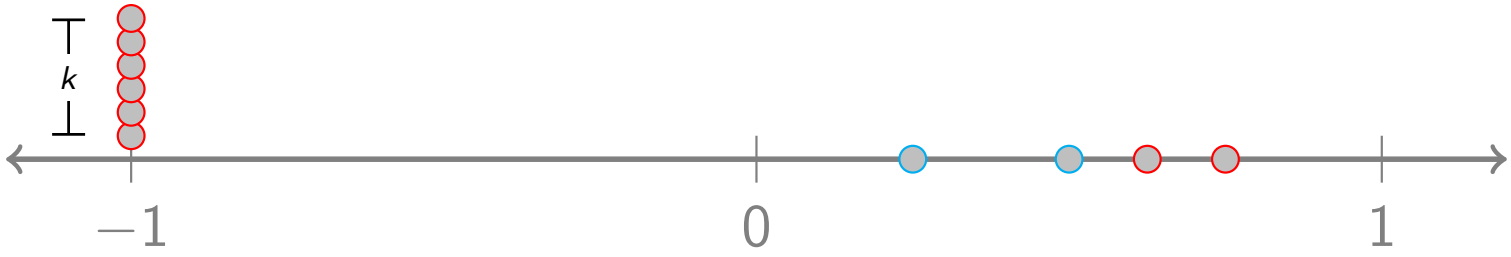

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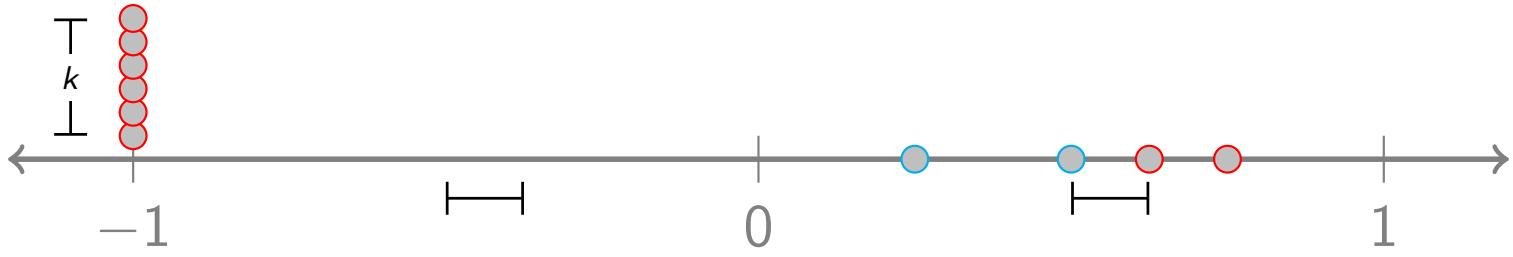

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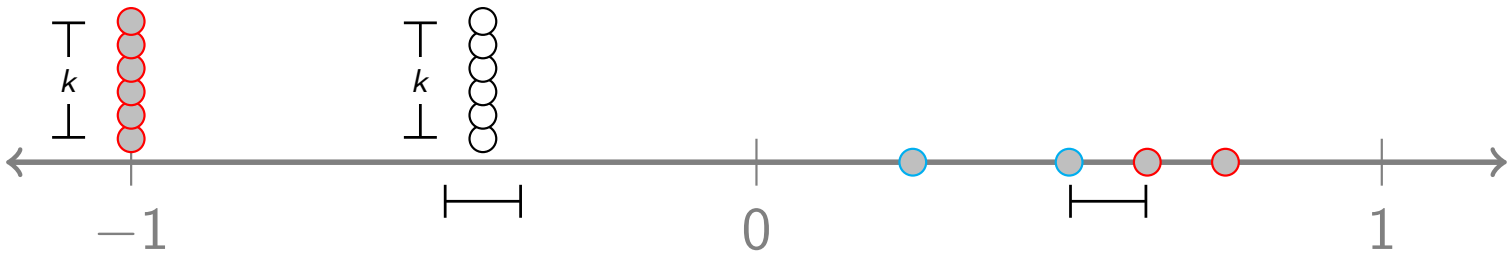

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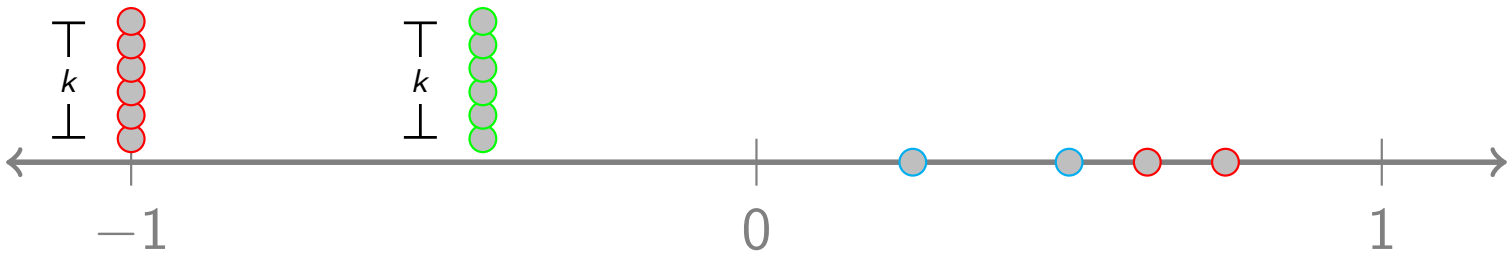

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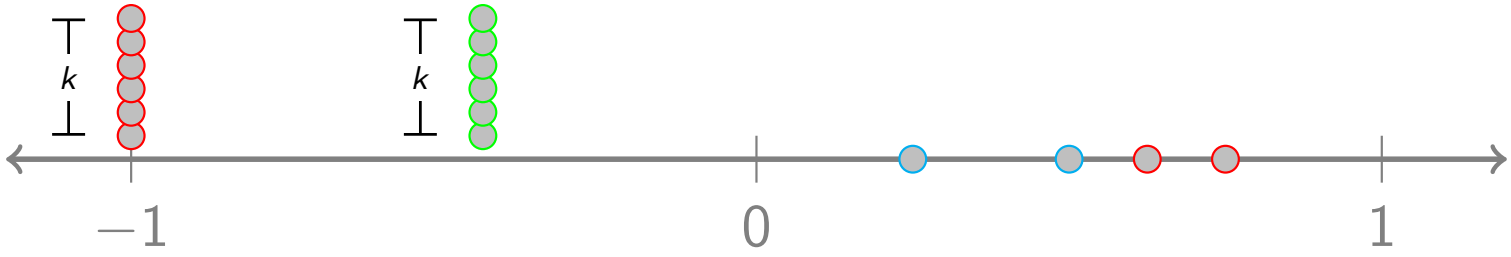

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- This forces $3k$ colors in a game of width $2k$.

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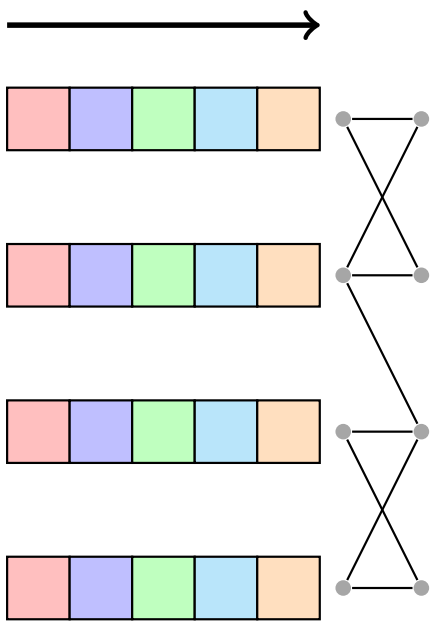

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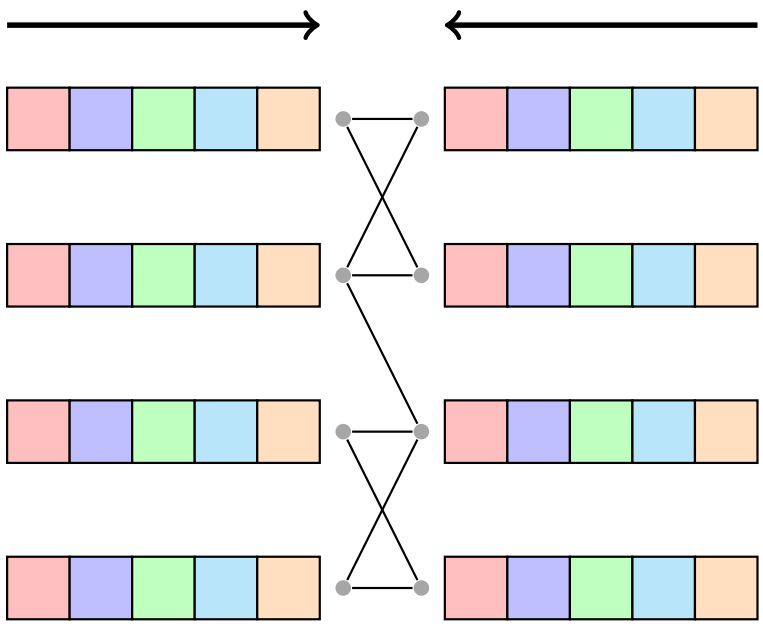

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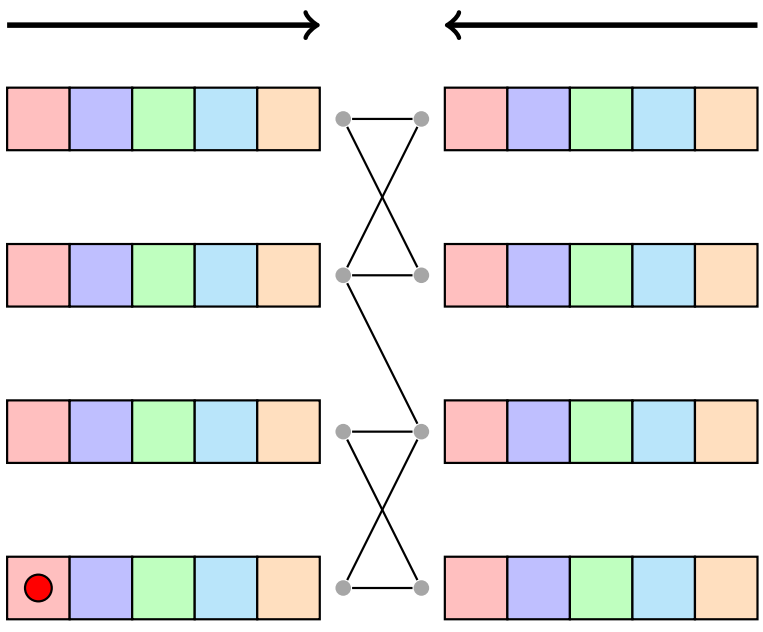

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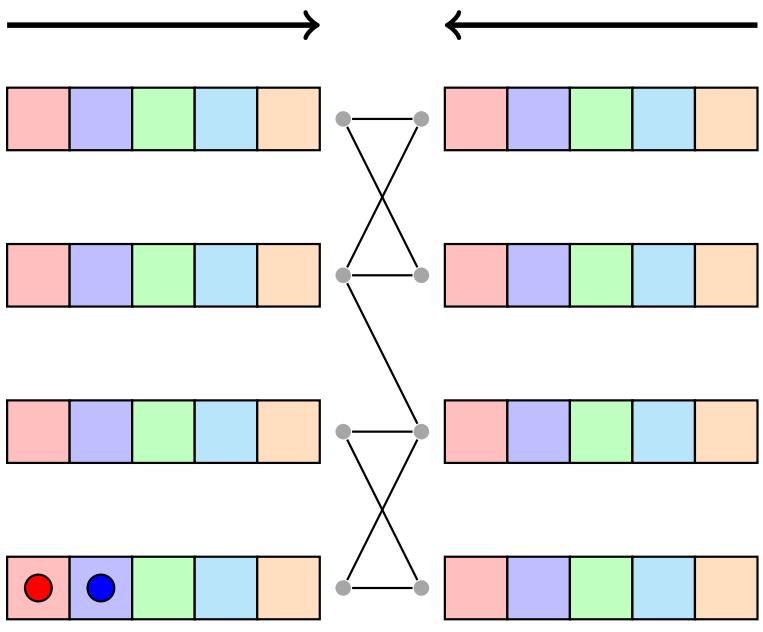

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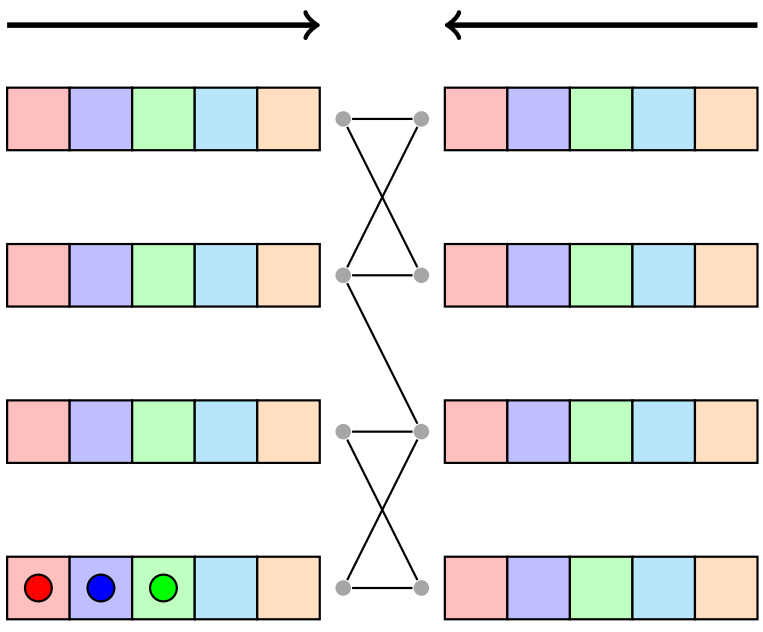
Bipartite graphs

- A graph G is online-perfect if $f(G; w) = w$.

Proposition

Every bipartite graph is online-perfect.



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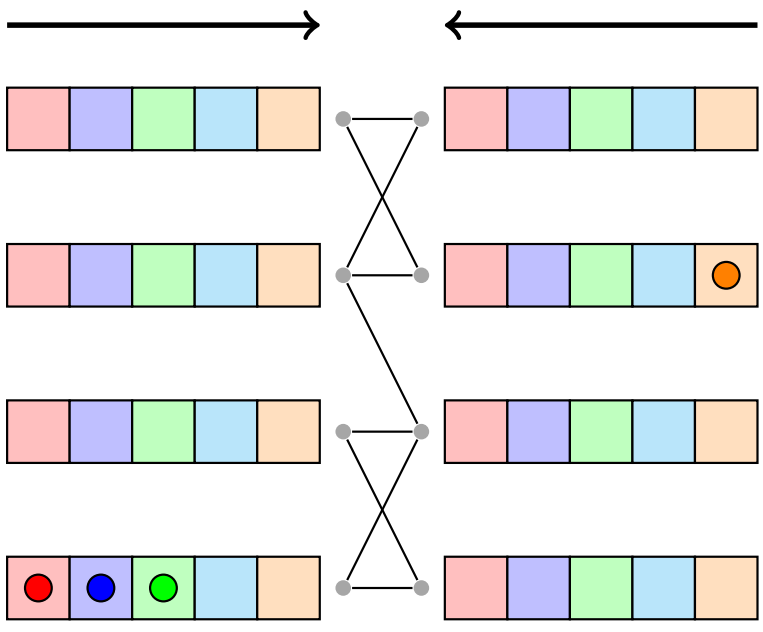

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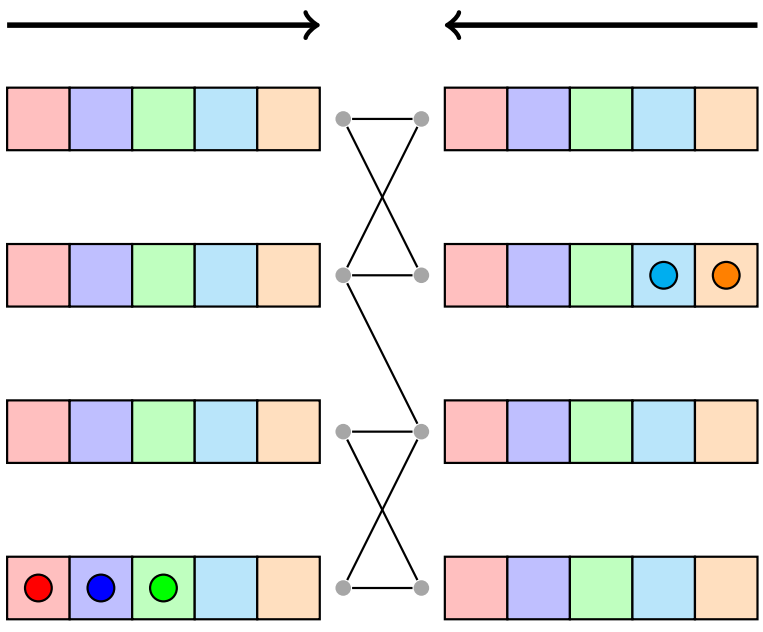

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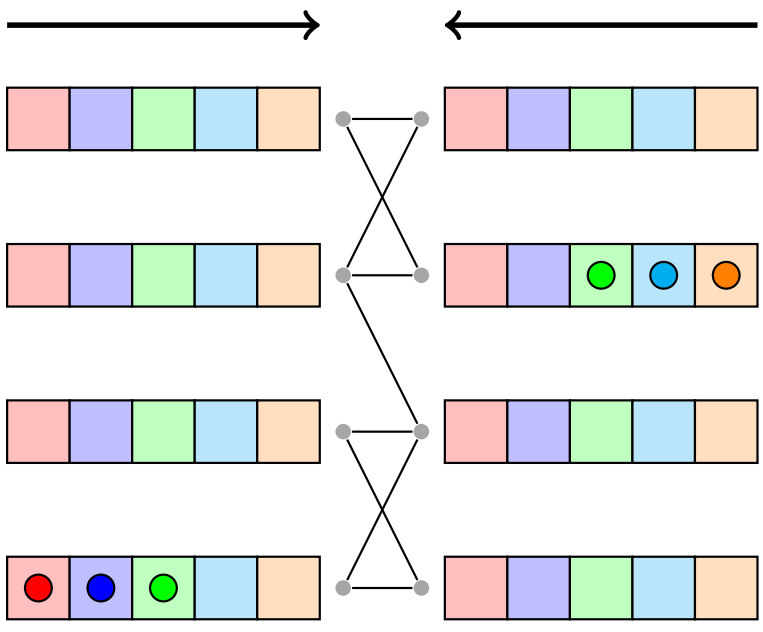

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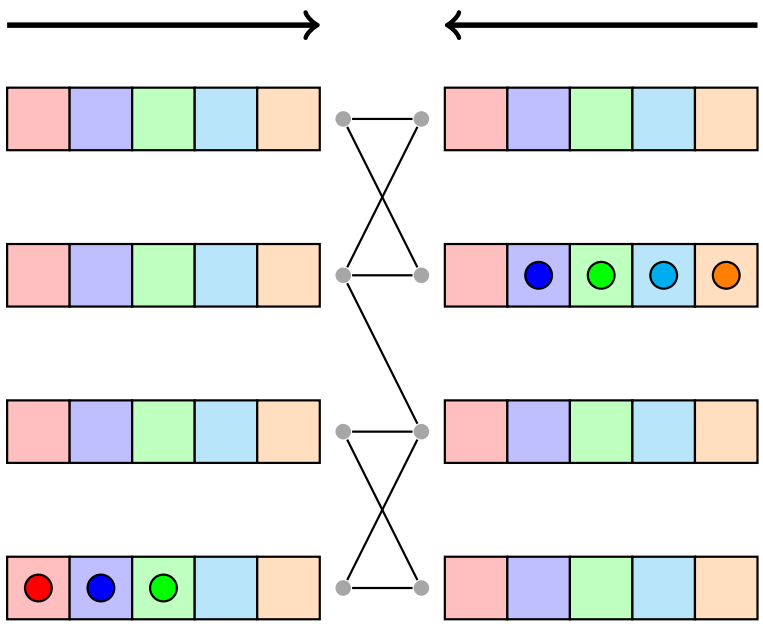

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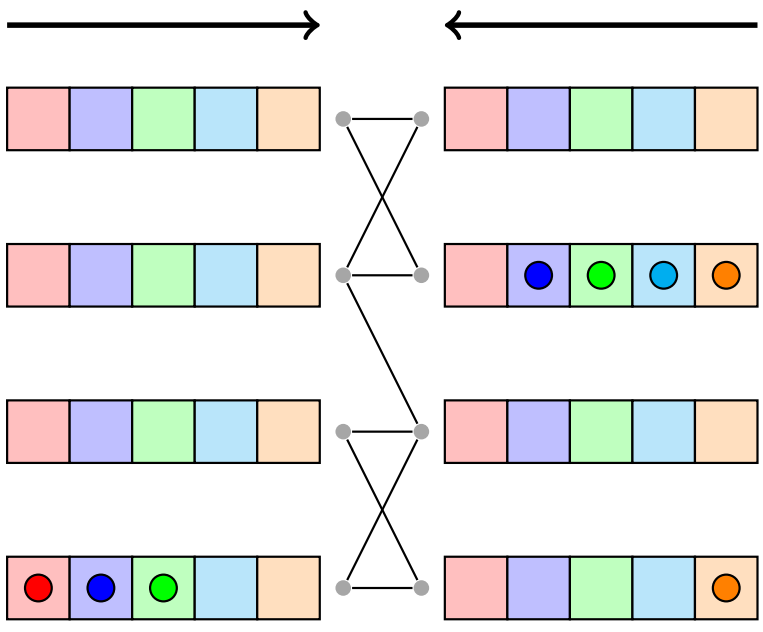

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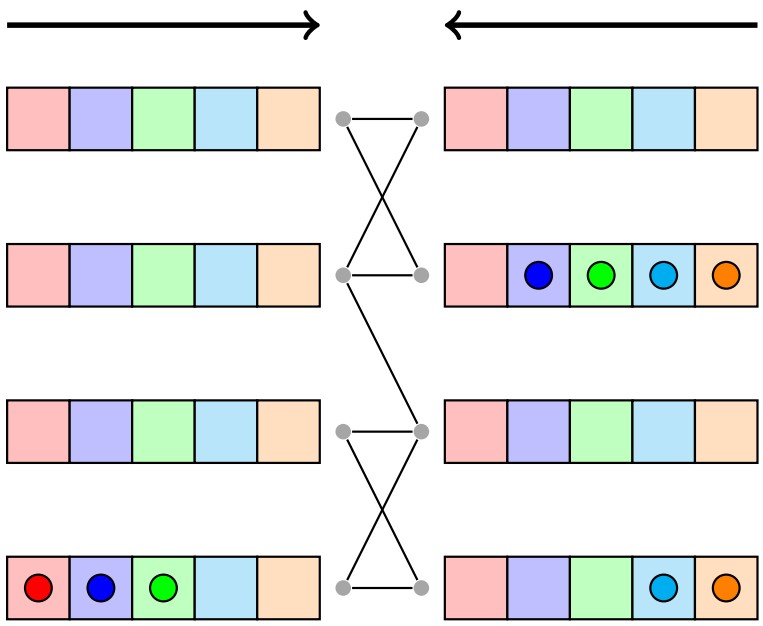

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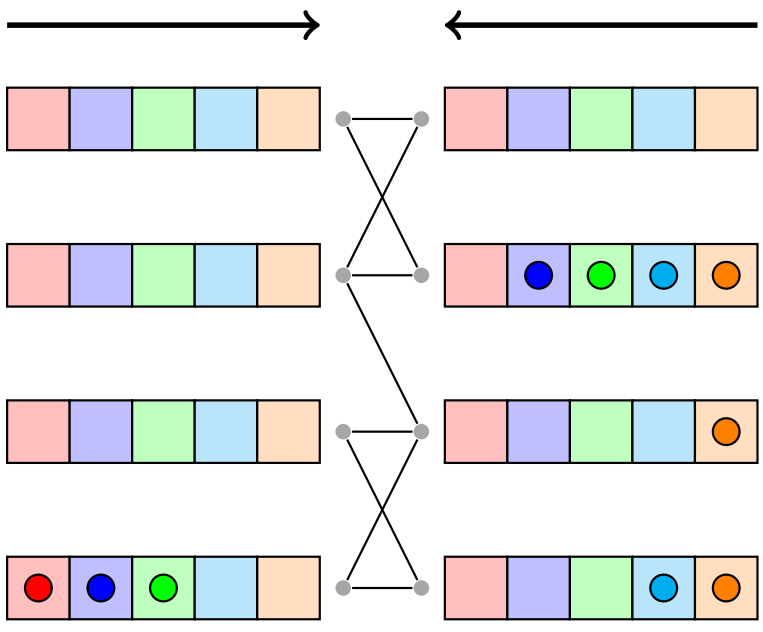

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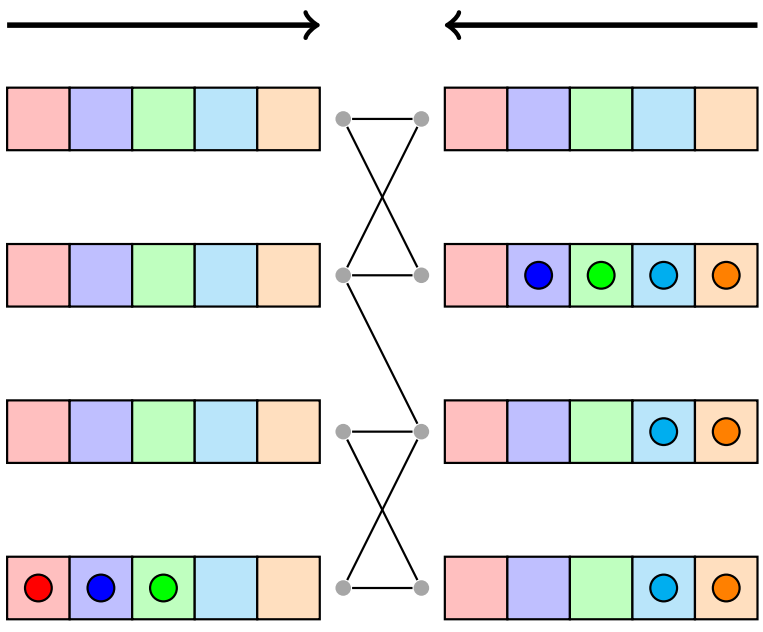

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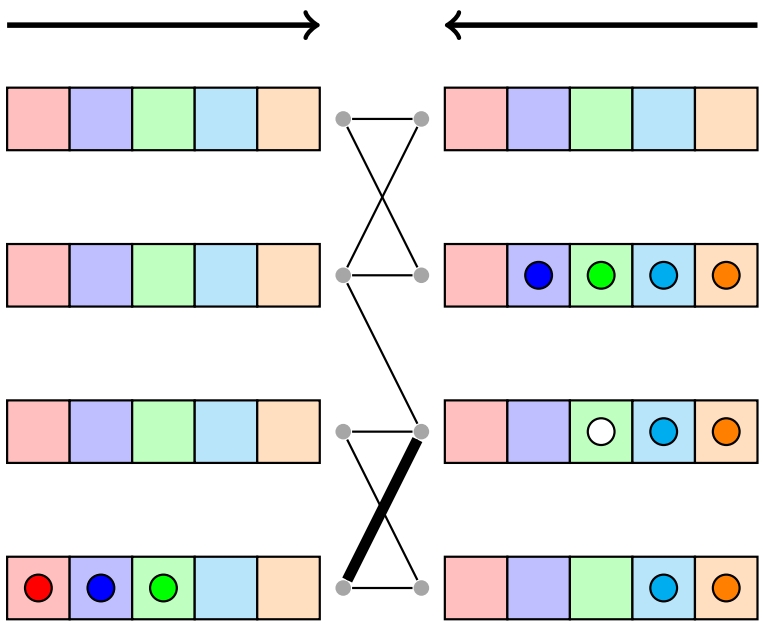

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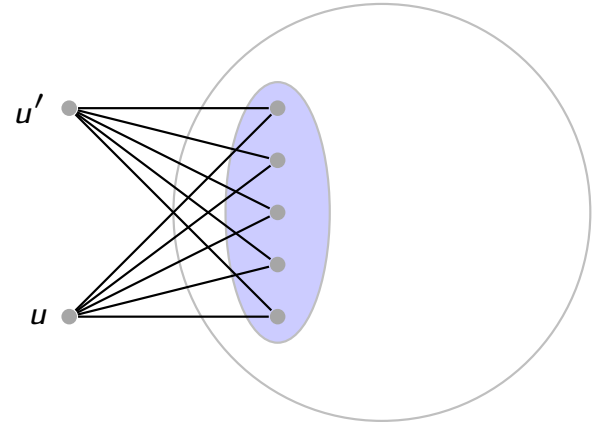
- A conflict would imply the token graph has a clique on more than w vertices.

Twins



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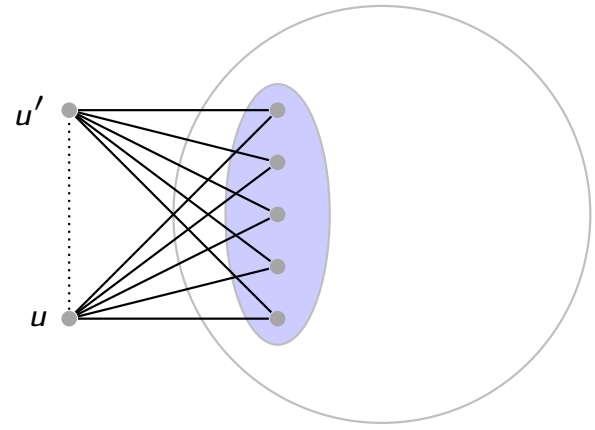

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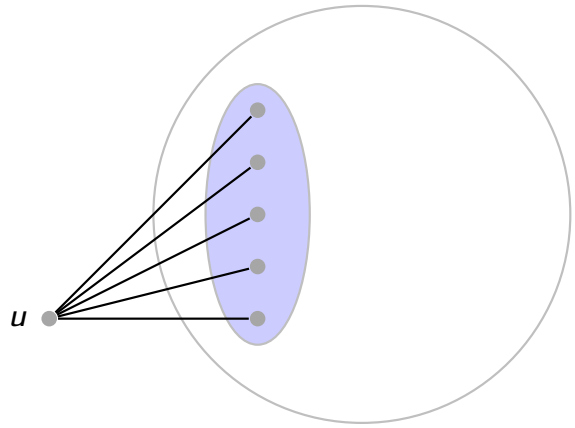

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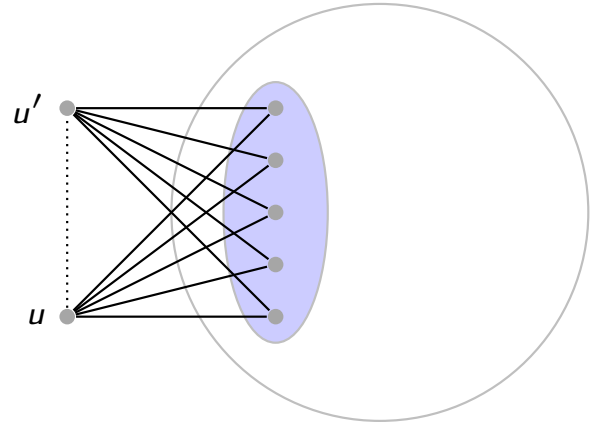

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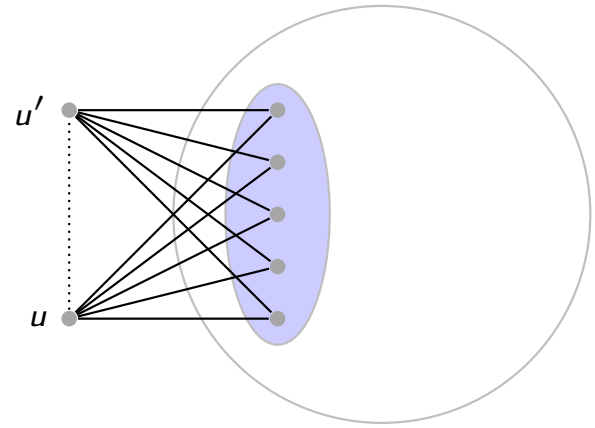

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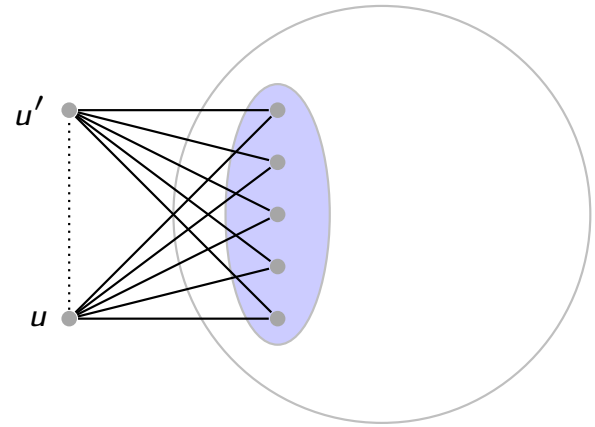
- Fact: a graph G is P_4-free if and only if G is obtainable from a single vertex by cloning.

Twins



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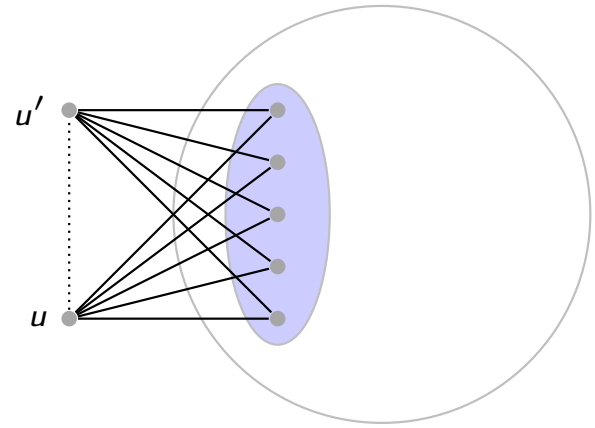

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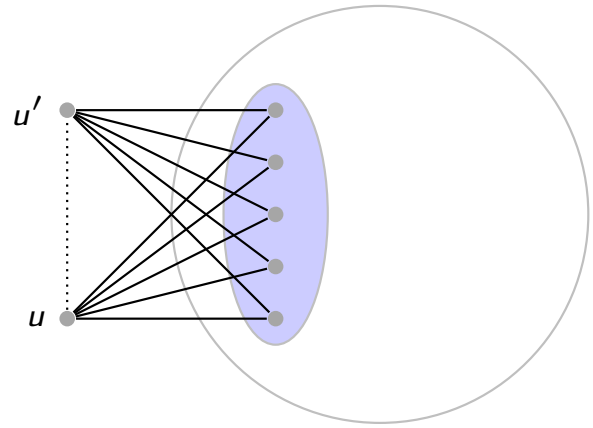

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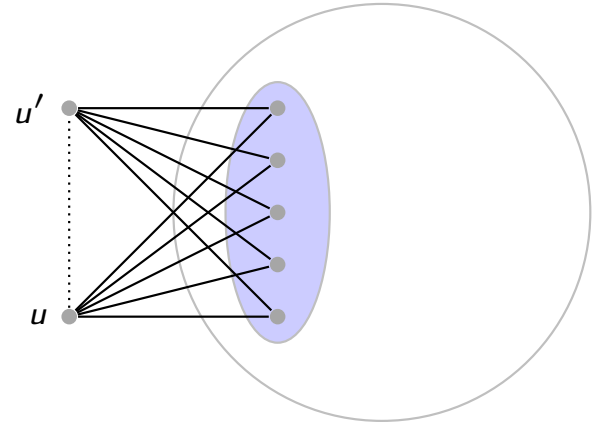

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Corollary

If G is obtainable from a bipartite graph by cloning, then G is online-perfect.

Characterization of online-perfect graphs

Theorem

Let G be a graph. The following are equivalent.

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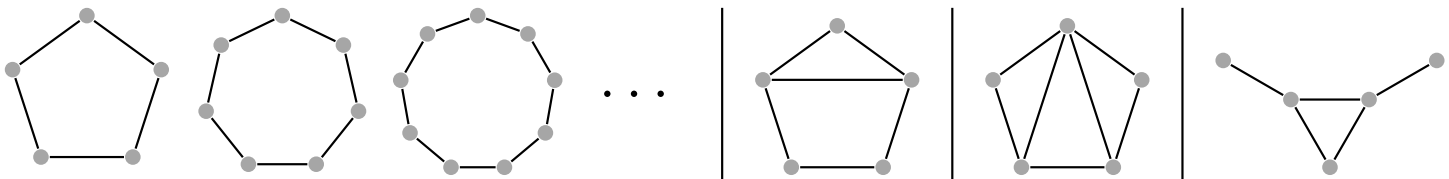

C_n for odd n at least 5 | C_5^+ | P_5^2 | *The bull*

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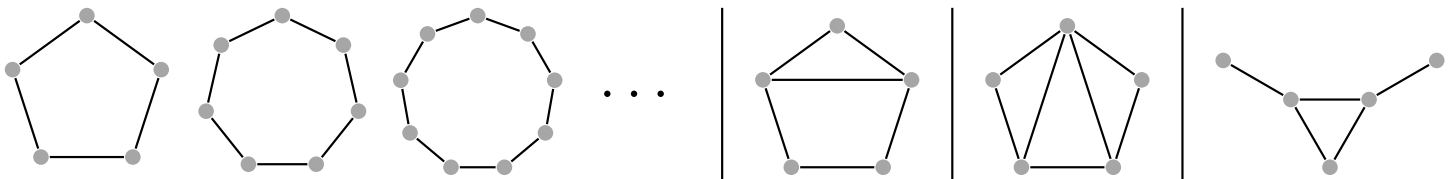

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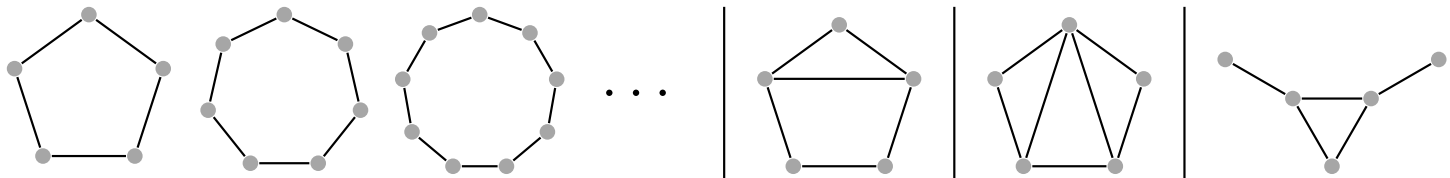

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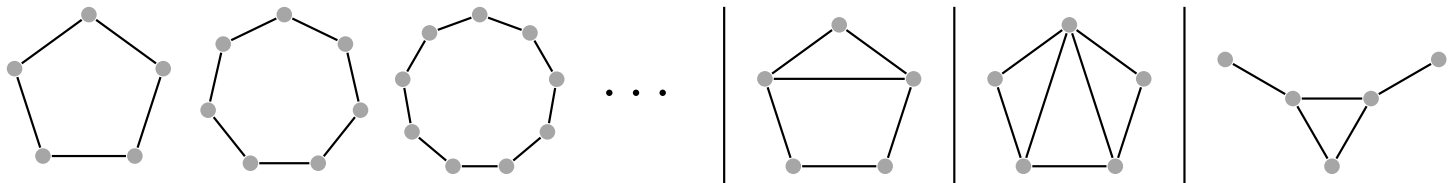

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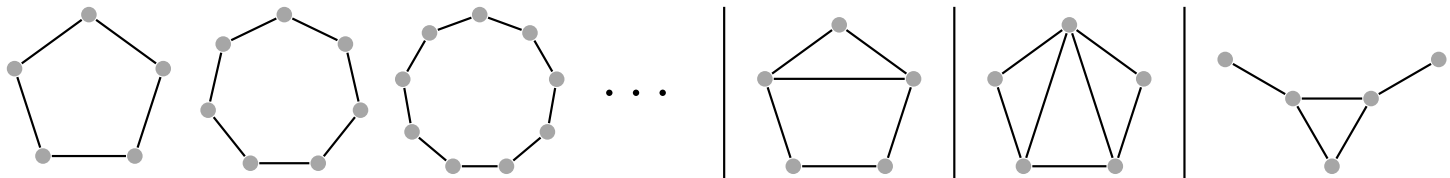

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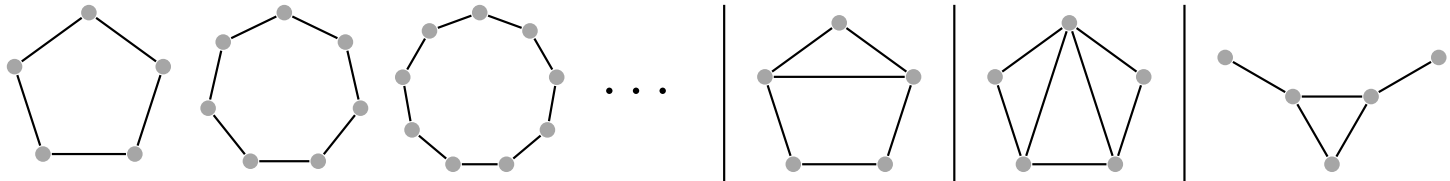

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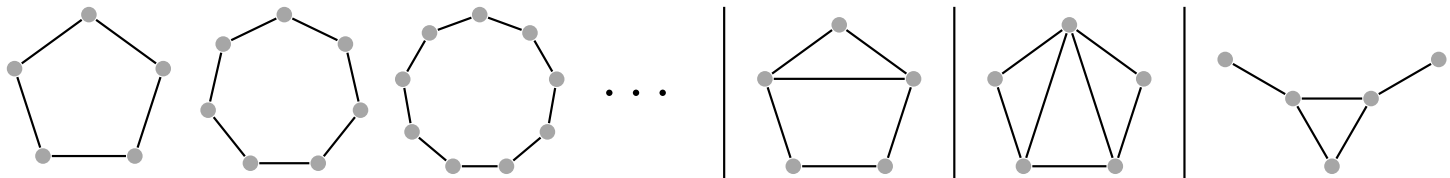

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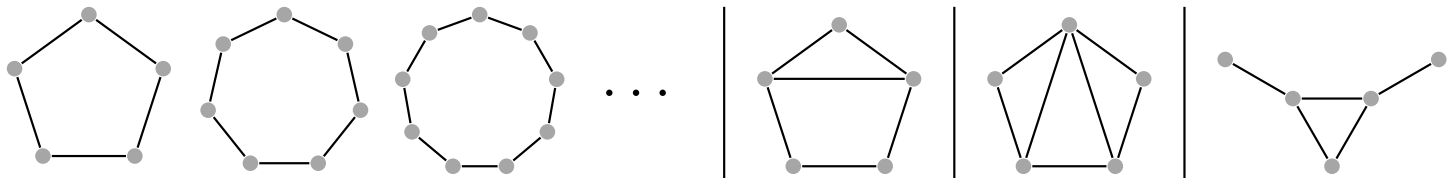

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- 4': G is online-perfect if and only if it the result of replacing each vertex in a bipartite graph with a P_4-free graph.

A Spoiler strategy

Lemma (Spoiler Lemma)

Let $U \subseteq V(G)$, where $U = \{u_1, \ldots, u_t\}$, and suppose that:

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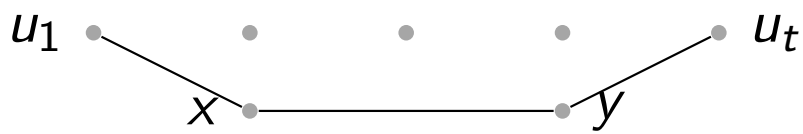

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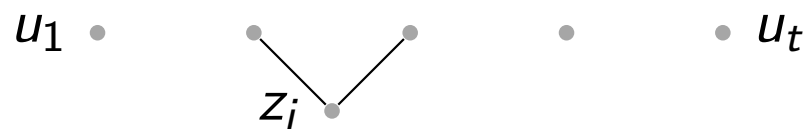

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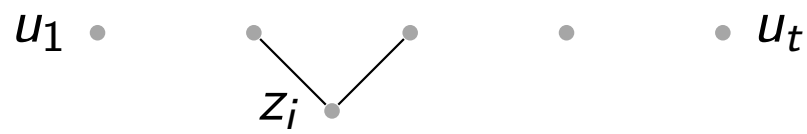

Minimal non-online-perfect graphs: lower bounds

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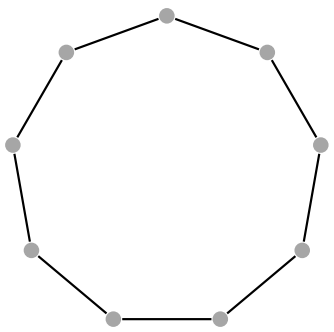

C_n

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If n is odd and $n \geq 5$ and w is even, then $f(C_n; w) \geq \frac{n}{n-1}w$.

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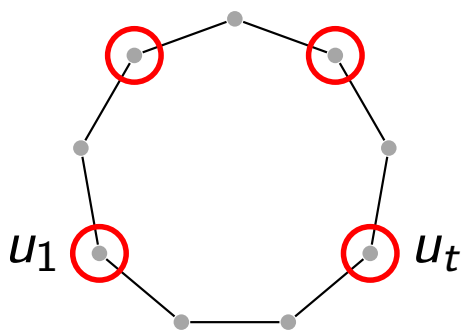

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- Apply Spoiler Lemma with $t = (n-1)/2$.

Minimal non-online-perfect graphs: lower bounds

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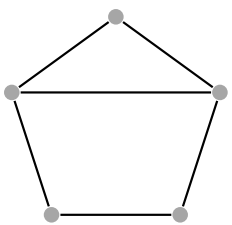
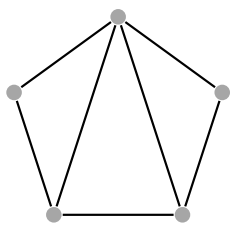
Let $U \subseteq V(G)$, where $U = \{u_1, \ldots, u_t\}$, and suppose that:

1. *There are vertices x and y such that u_1xyu_t is a path and $G[\{u_1, x, y, u_t\}]$ is bipartite, and*
2. *For each i, there is a common neighbor z_i of u_i and u_{i+1} such that $G[U \cup \{z_i\}]$ is bipartite.*

If w is even, then $f(G; w) \geq (1 + \frac{1}{2t})w$.



C_5^+



P_5^2

Corollary

If $G \in \{C_5^+, P_5^2\}$ and w is even, then $f(G; w) \geq \frac{5}{4}w$.

Minimal non-online-perfect graphs: lower bounds

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Let $U \subseteq V(G)$, where $U = \{u_1, \ldots, u_t\}$, and suppose that:

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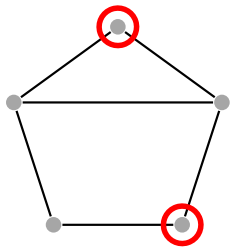

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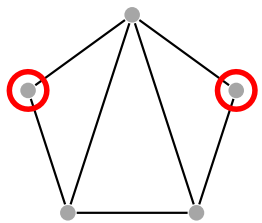

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Corollary

If $G \in \{C_5^+, P_5^2\}$ and w is even, then $f(G; w) \geq \frac{5}{4}w$.

- Apply Spoiler Lemma with $t = 2$.

Minimal non-online-perfect graphs: lower bounds

Lemma (Spoiler Lemma)

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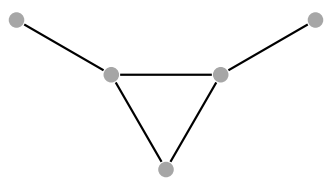

Corollary

If G is the bull graph and w is even, then $f(G; w) \geq \frac{7}{6}w$.

Minimal non-online-perfect graphs: lower bounds

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Let $U \subseteq V(G)$, where $U = \{u_1, \ldots, u_t\}$, and suppose that:

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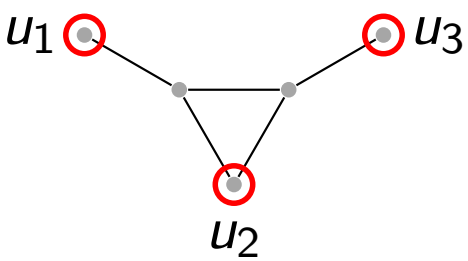

Corollary

If G is the bull graph and w is even, then $f(G; w) \geq \frac{7}{6}w$.

- Apply Spoiler Lemma with $t = 3$.

Minimal non-online-perfect graphs: upper bounds

Proposition (Fractional Coloring Strategy)

If G has a (p, q)-coloring, then $f(G; w) \leq p \left\lceil \frac{w}{2q} \right\rceil$. In particular, $f(G; w) \leq (\frac{1}{2}\chi_f(G) + o(1))w$.

Minimal non-online-perfect graphs: upper bounds

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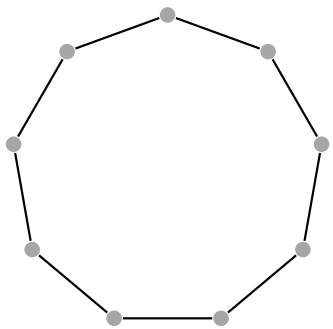


C_n

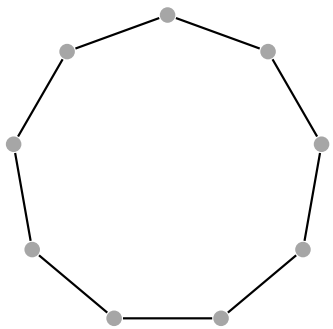
Corollary

If n is odd, then $f(C_n; w) \leq (\frac{n}{n-1} + o(1))w$.

Minimal non-online-perfect graphs: upper bounds

Proposition (Fractional Coloring Strategy)

If G has a (p, q)-coloring, then $f(G; w) \leq p \left\lceil \frac{w}{2q} \right\rceil$. In particular, $f(G; w) \leq (\frac{1}{2}\chi_f(G) + o(1))w$.



C_n

Corollary

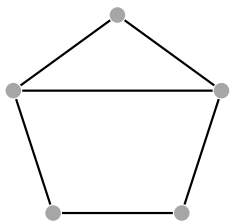
If n is odd, then $f(C_n; w) \leq (\frac{n}{n-1} + o(1))w$.

- Apply Prop. with $\chi_f(C_n) = \frac{2n}{n-1}$.

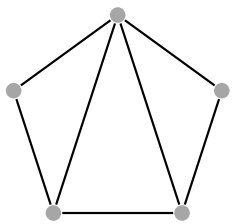
Minimal non-online-perfect graphs: upper bounds

Proposition (Fractional Coloring Strategy)

If G has a (p, q)-coloring, then $f(G; w) \leq p \left\lceil \frac{w}{2q} \right\rceil$. In particular, $f(G; w) \leq (\frac{1}{2}\chi_f(G) + o(1))w$.



C_5^+



P_5^2

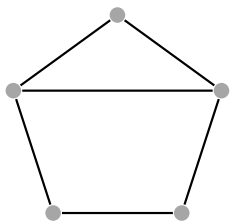
Corollary

If $G \in \{C_5^+, P_5^2\}$, *then* $f(G; w) \leq (\frac{3}{2} + o(1))w$.

Minimal non-online-perfect graphs: upper bounds

Proposition (Fractional Coloring Strategy)

If G has a (p, q)-coloring, then $f(G; w) \leq p \left\lceil \frac{w}{2q} \right\rceil$. In particular, $f(G; w) \leq (\frac{1}{2}\chi_f(G) + o(1))w$.



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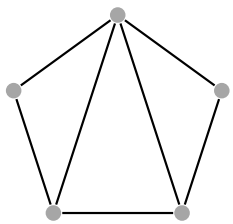

P_5^2

Corollary

If $G \in \{C_5^+, P_5^2\}$, *then* $f(G; w) \leq (\frac{3}{2} + o(1))w$.

- Apply Prop. with $\chi_f(G) = \chi(G) = 3$.

Minimal non-online-perfect graphs: upper bounds

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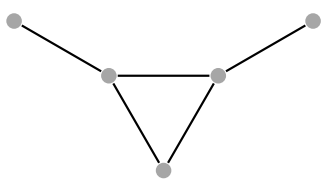

Corollary

If G is the bull graph and w is even, then $f(G; w) \leq (\frac{3}{2} + o(1))w$.

Minimal non-online-perfect graphs: upper bounds

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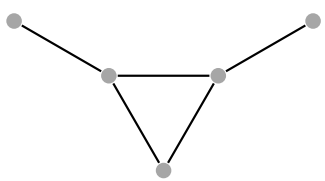
If G has a (p, q)-coloring, then $f(G; w) \leq p \left\lceil \frac{w}{2q} \right\rceil$. In particular, $f(G; w) \leq (\frac{1}{2}\chi_f(G) + o(1))w$.



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If G is the bull graph and w is even, then $f(G; w) \leq (\frac{3}{2} + o(1))w$.

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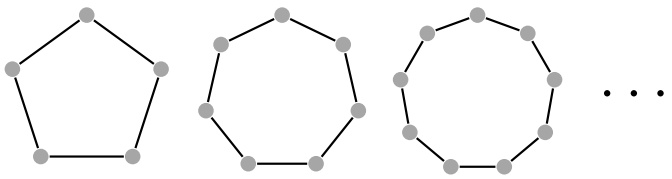
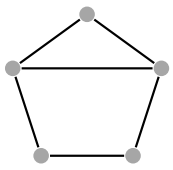
Summary

Theorem

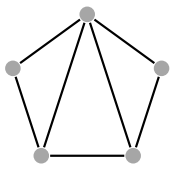
A graph is online-perfect if and only if it does not have an induced copy of any of the following:



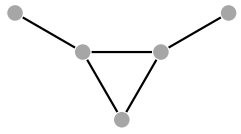
C_n *for odd* n *at least* 5



C_5^+



P_5^2



The bull

Summary

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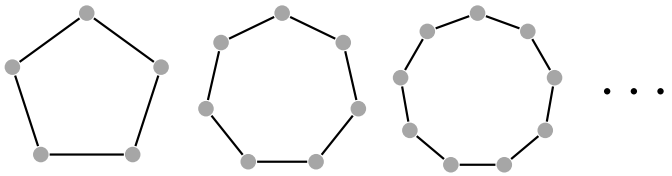

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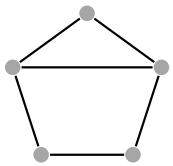

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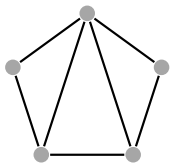

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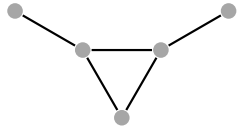

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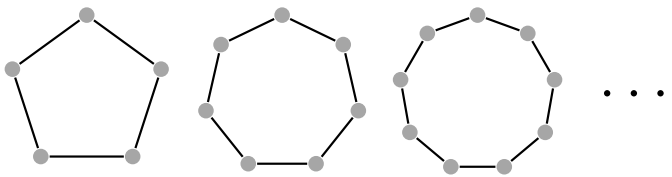

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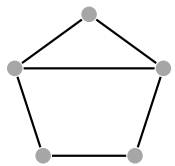

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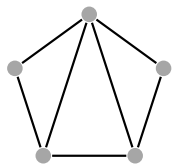

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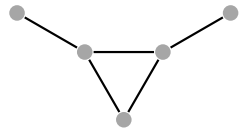

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- For odd n and $n \geq 5$, we have $f(C_n; w) = (\frac{n}{n-1} + o(1))w$.
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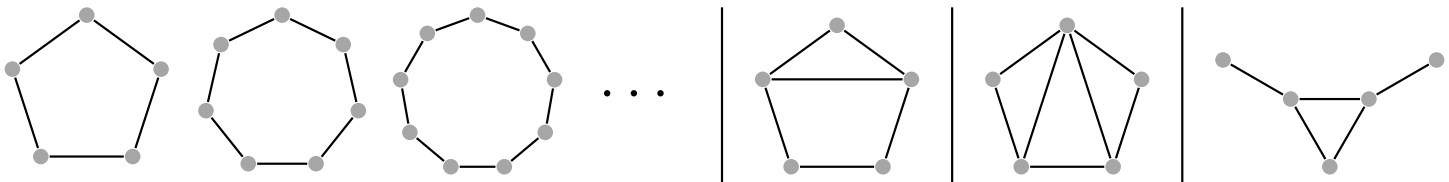
A graph is online-perfect if and only if it does not have an induced copy of any of the following:



C_n *for odd* n *at least* 5 | C_5^+ | P_5^2 | *The bull*

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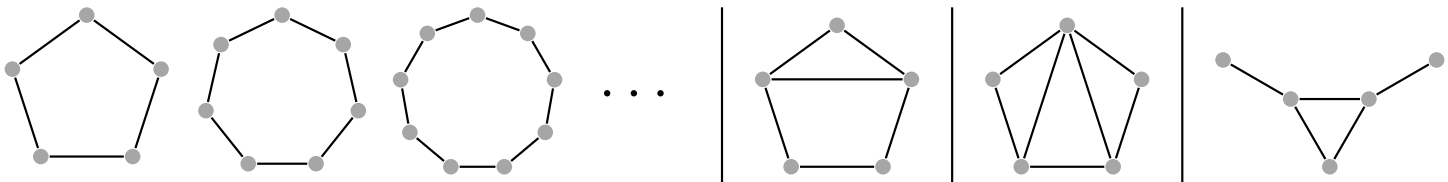

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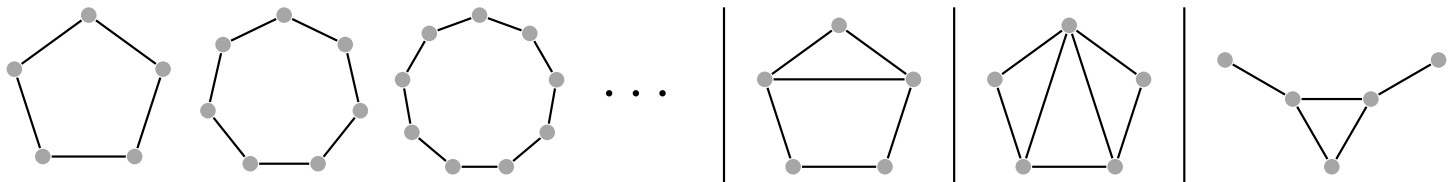

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Open Problems

Determine the asymptotics of $f(C_5^+; w)$,

Summary

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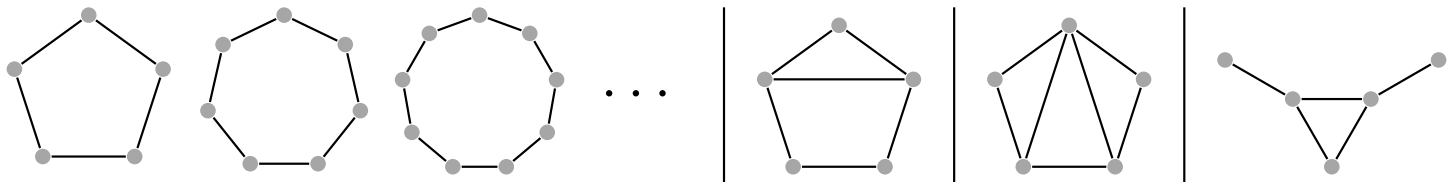

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Determine the asymptotics of $f(C_5^+; w)$, $f(B; w)$,

Summary

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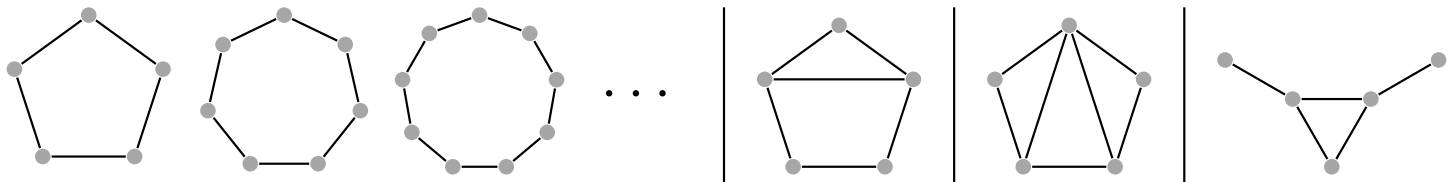

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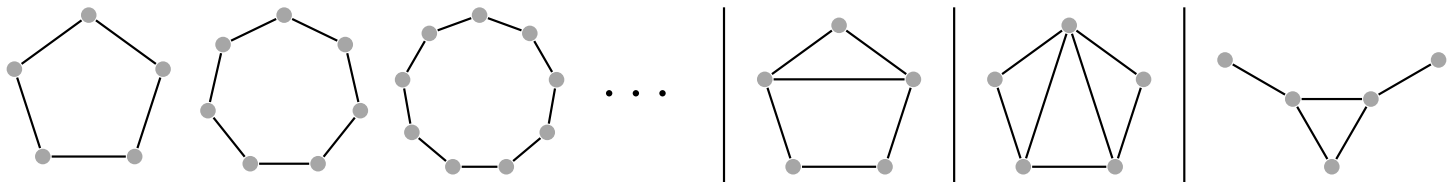
Open Problems

Determine the asymptotics of $f(C_5^+; w)$, $f(B; w)$, $f(\mathbb{G}; w)$.

Summary

Theorem

A graph is online-perfect if and only if it does not have an induced copy of any of the following:



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Thank You.