Coloring Clean and K_4-free Circle Graphs

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Circle Graphs

Definition

A circle graph is the intersection graph of chords in a circle.

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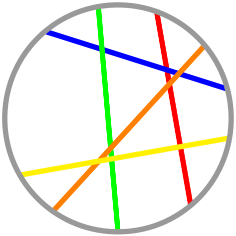

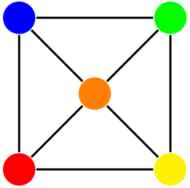

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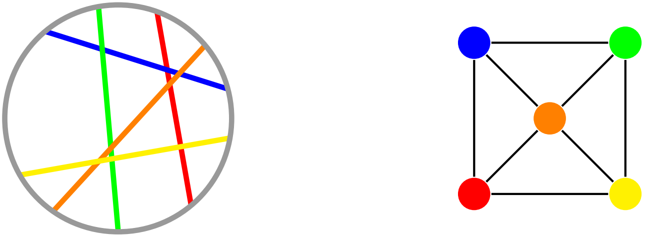

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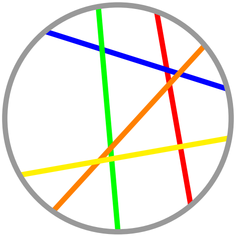

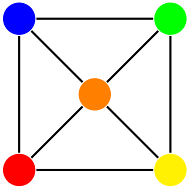

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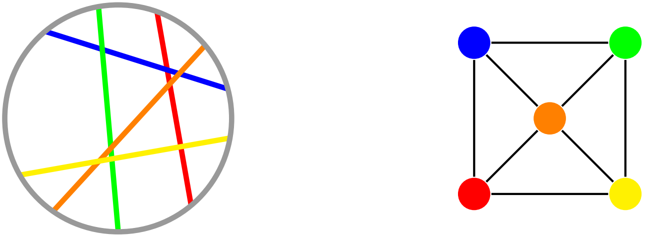

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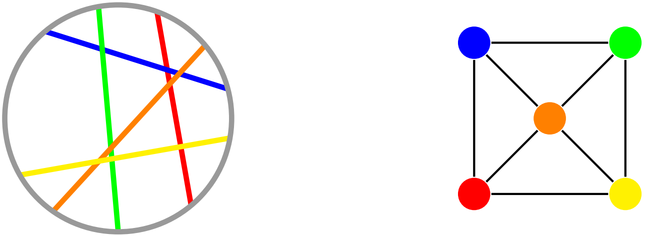

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- questions in topological graph theory,
- and VLSI design.
- We may assume the endpoints are distinct.

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For infinitely many k, there is a circle graph G with $\omega(G) = k$ and

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- This exponential gap has persisted for 25 years.

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- Sharp bounds are known only for triangle-free circle graphs.

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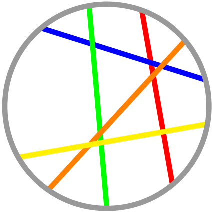

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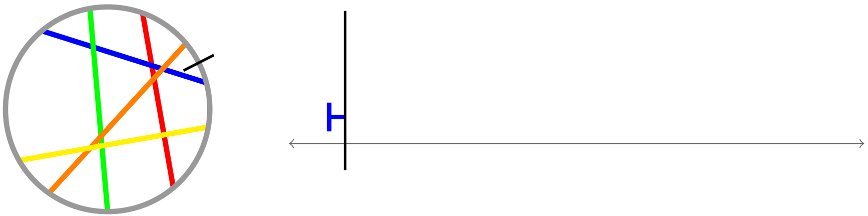

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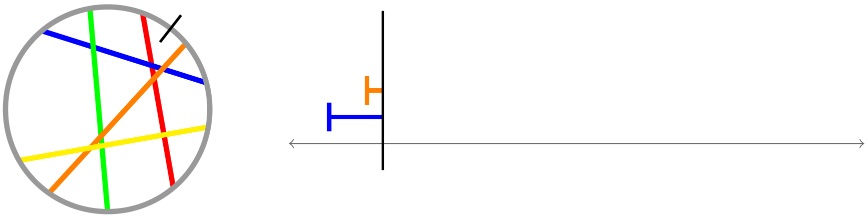

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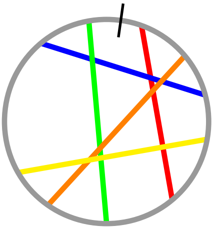

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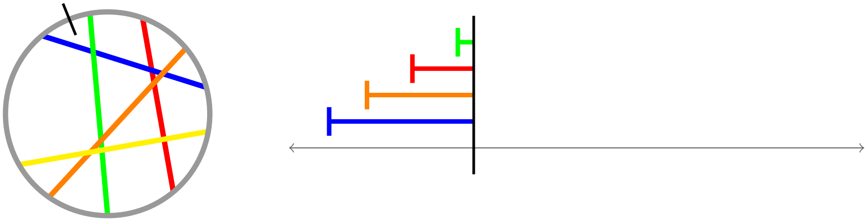

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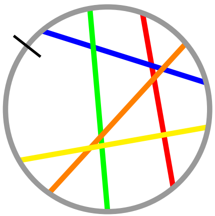

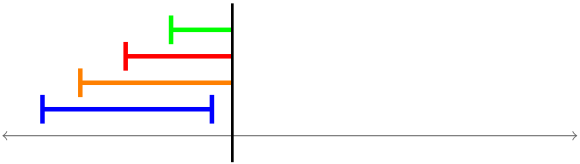

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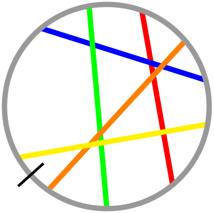

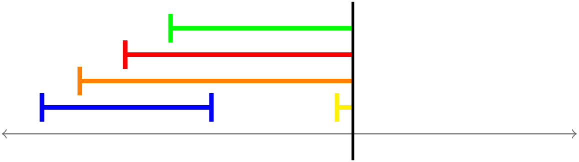

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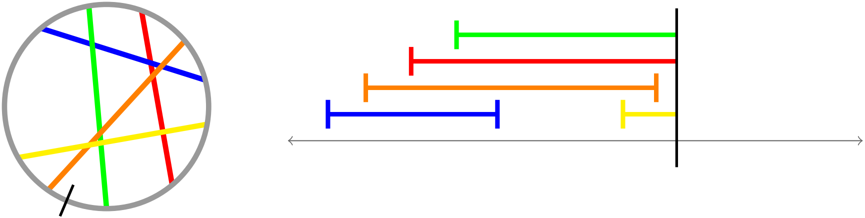

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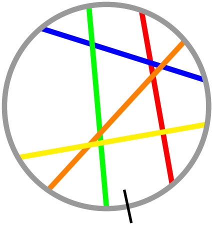

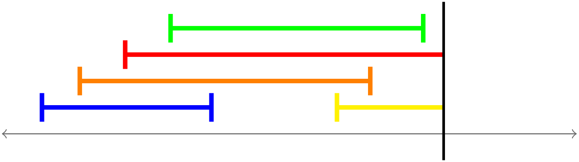

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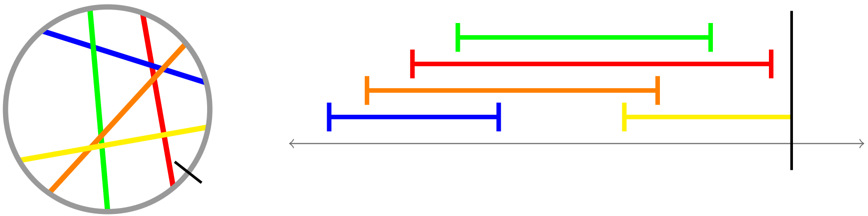

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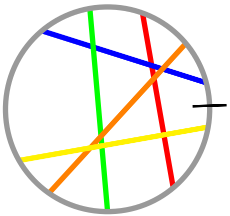

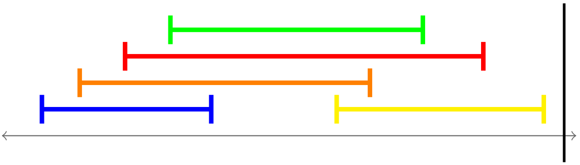

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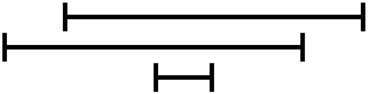

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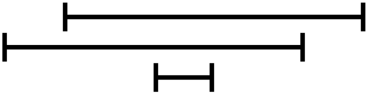

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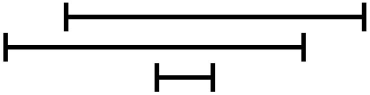

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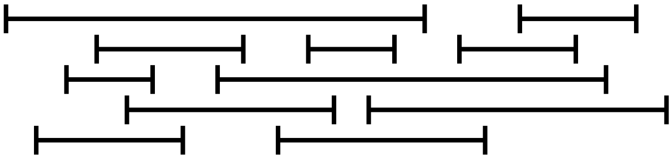

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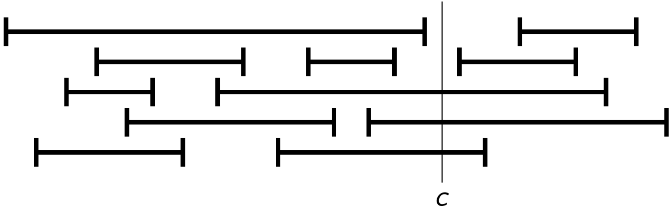

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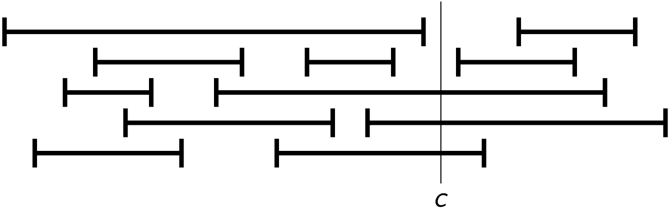

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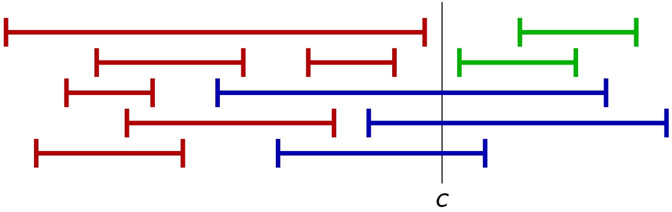

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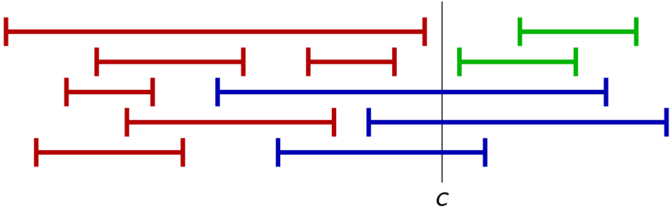

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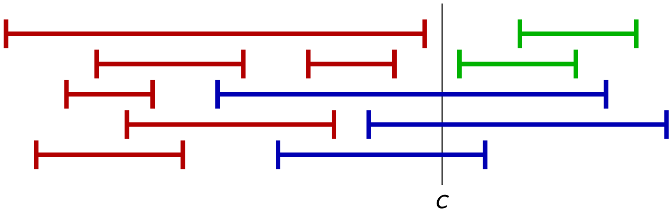

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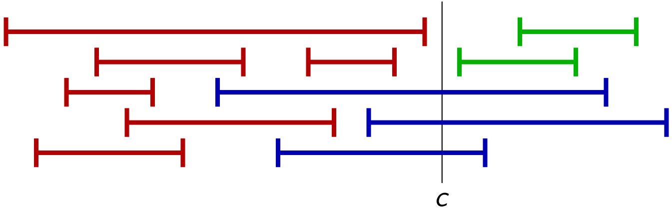

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- We need to control the coloring on X^{c}.

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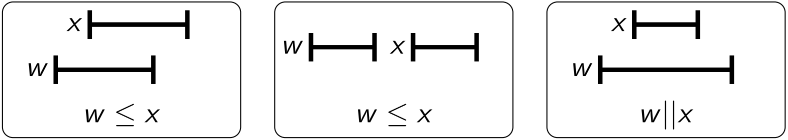

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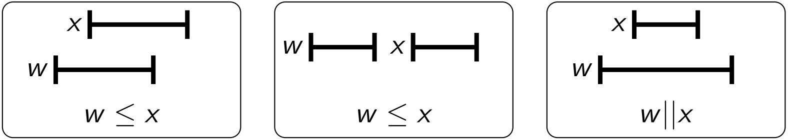

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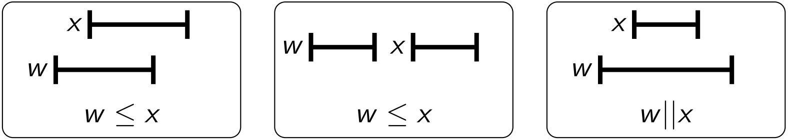

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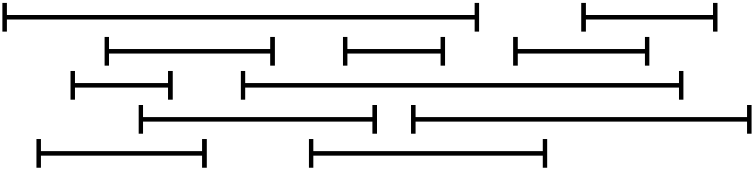

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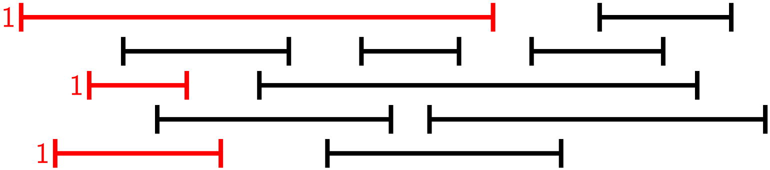

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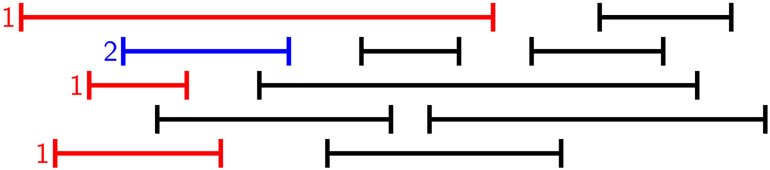

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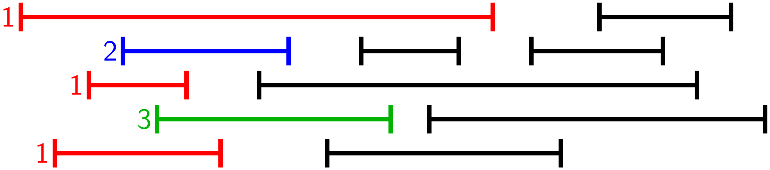

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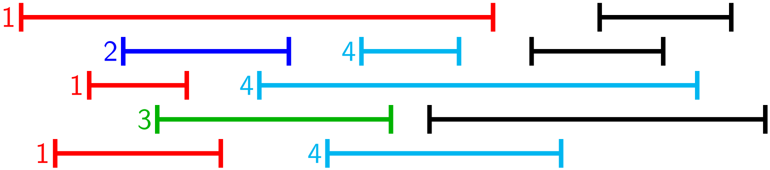

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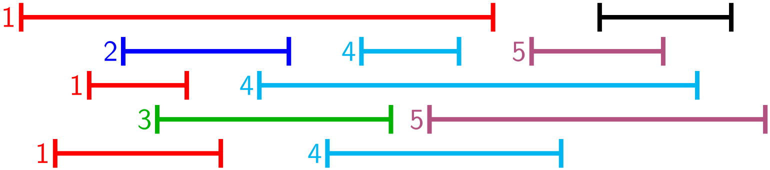

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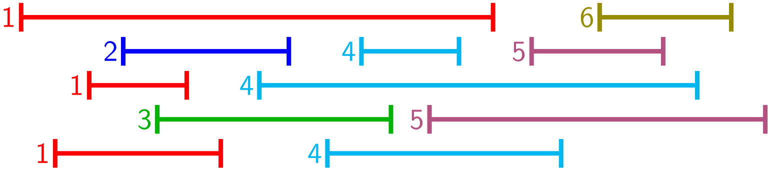

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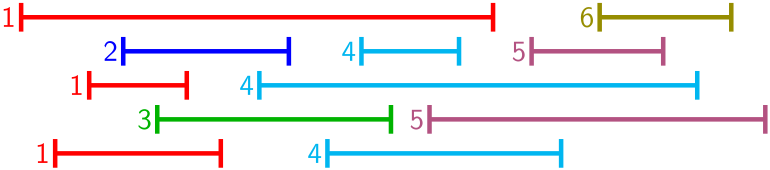

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Proposition

When all intervals share a common point, the overlap graph is perfect and the canonical coloring is optimal.

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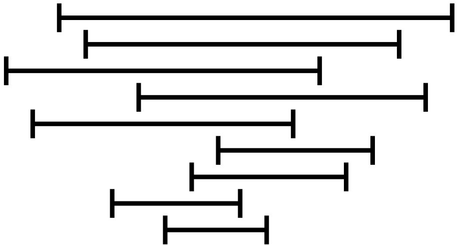

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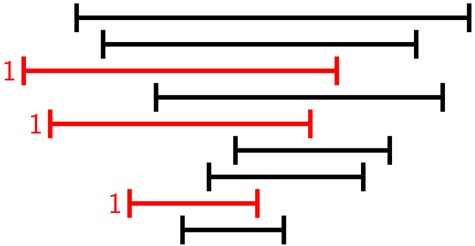

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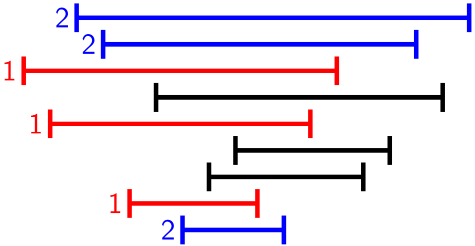

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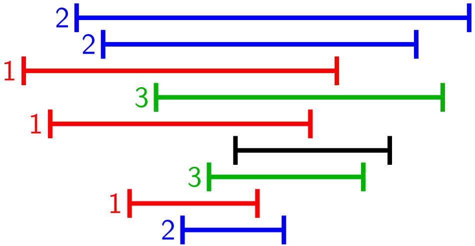

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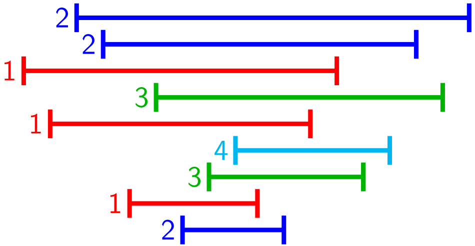

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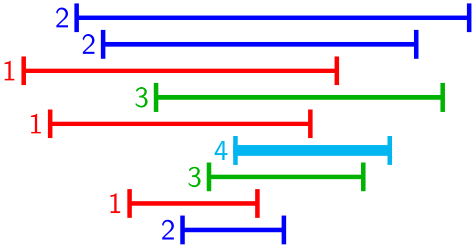

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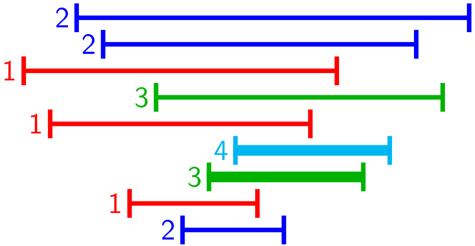

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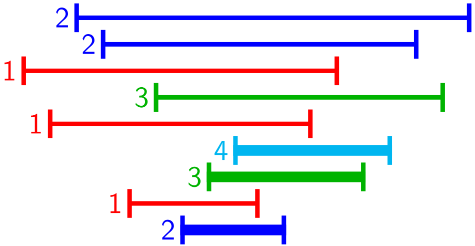

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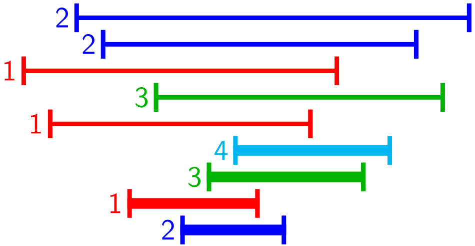

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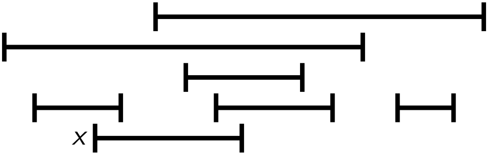
Good Colorings



- The closed right-neighborhood of x, denoted $R[x]$, consists of x and all neighbors of x that overlap x to the right.

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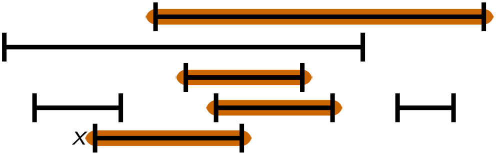

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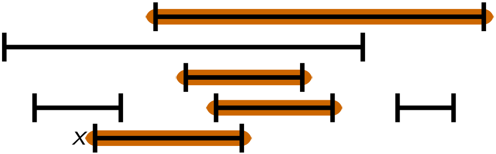

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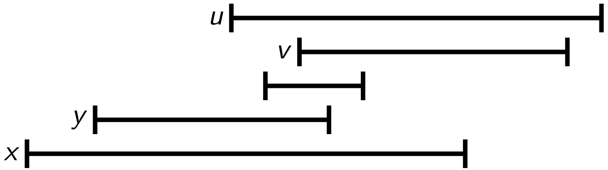

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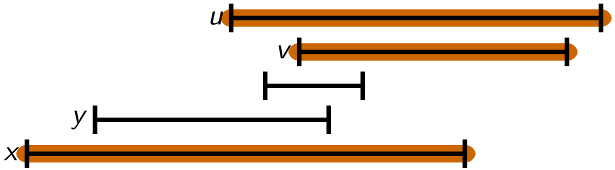

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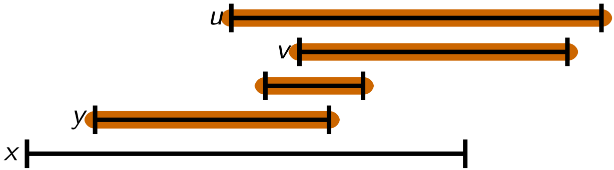

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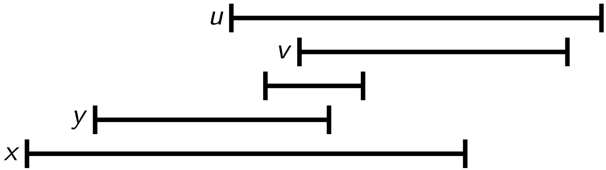

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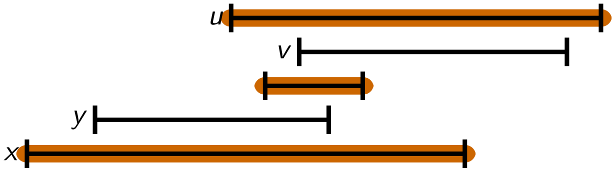

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If X is a clean family of intervals whose overlap graph has clique number k, then X has a good coloring using at most $2k-1$ colors.

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- $k = 3$: 5 colors are sufficient and 4 are sometimes necessary.

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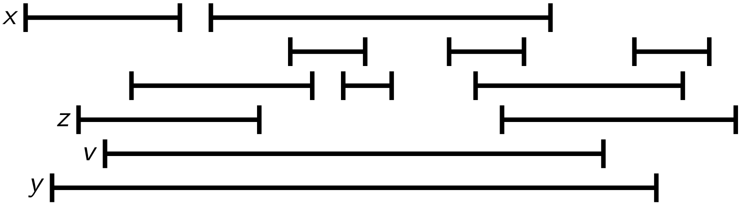
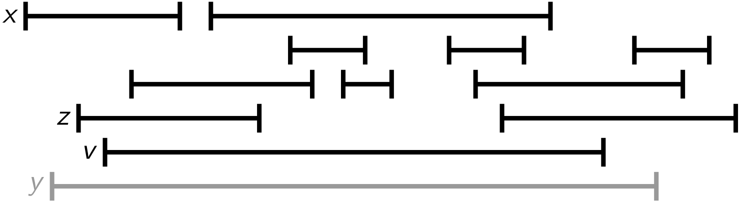


$k = 3$; colors: 1 2 3 4 5

- Find appropriate intervals x, y, z, and v. Delete y.

Coloring Clean Families



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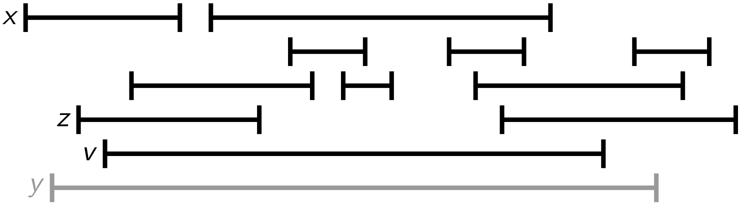

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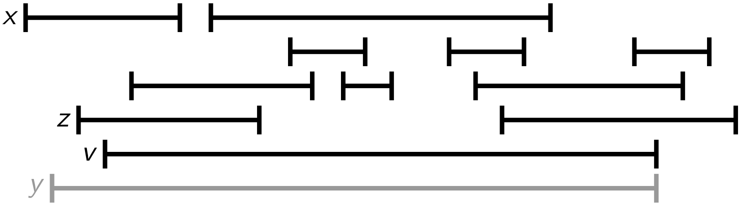

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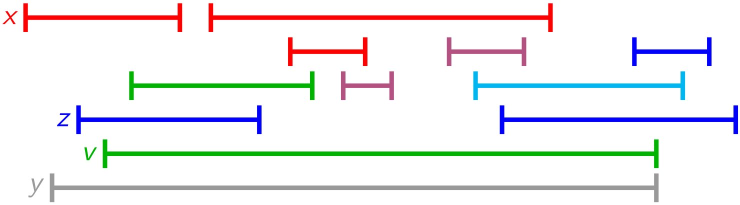

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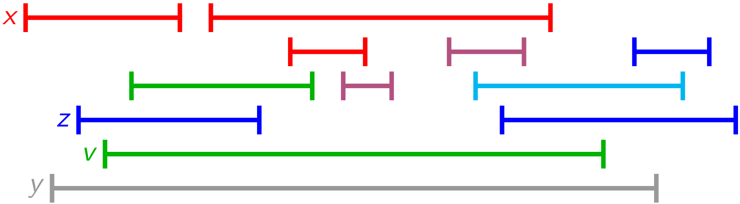

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- Find appropriate intervals x, y, z, and v. Delete y.
- With y removed, z and v are inclusion-maximal.
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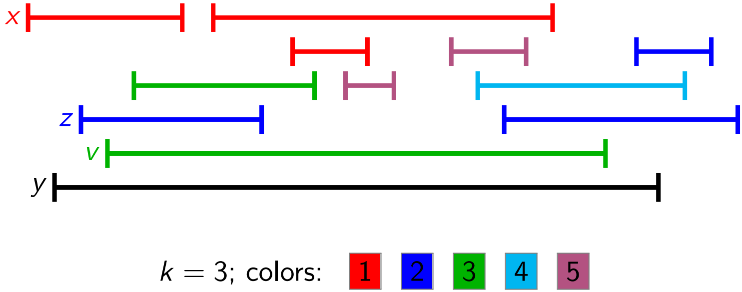

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- Restore y.

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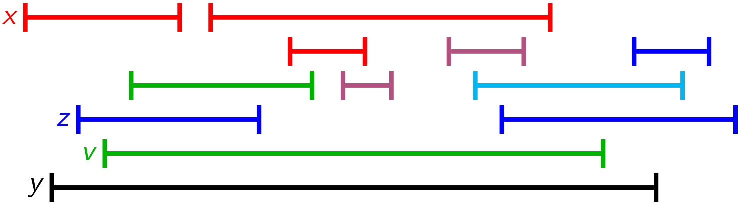

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- How should we color y?

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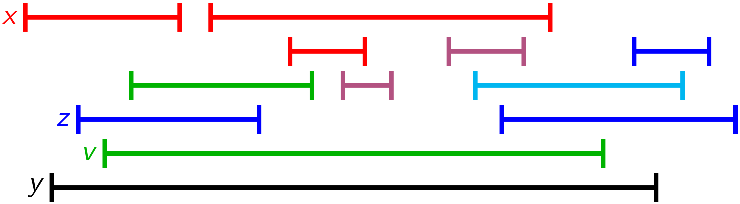

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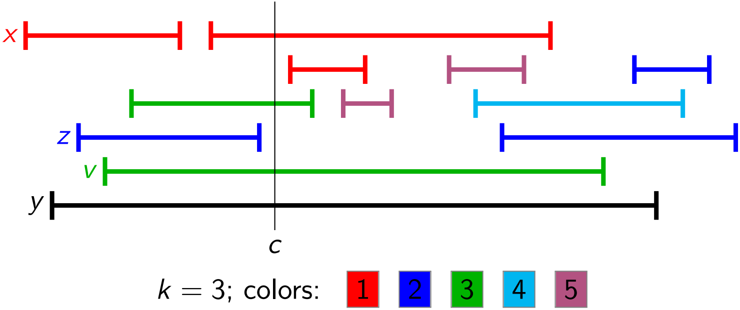

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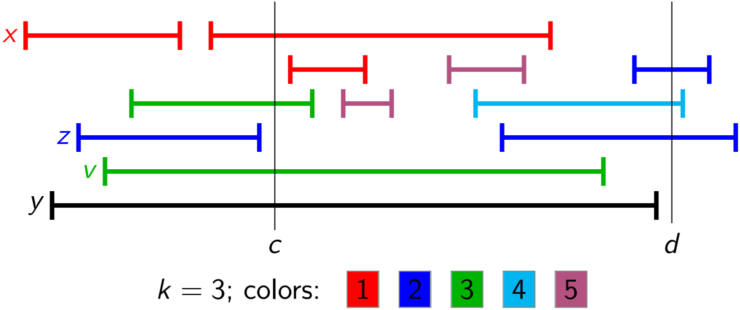

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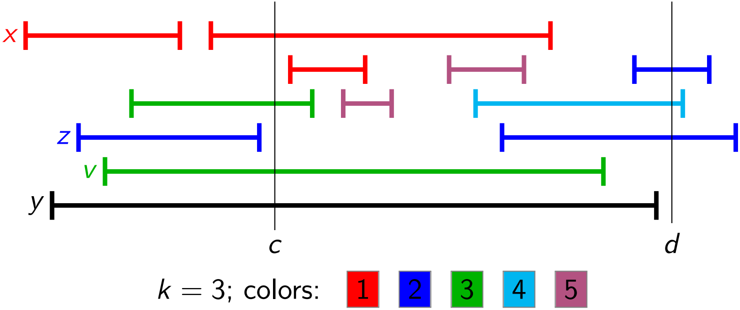

- But z and v were inclusion-maximal.

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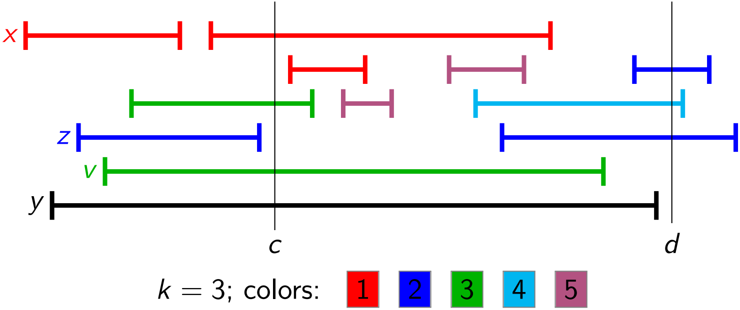

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- The coloring is canonical on X^c and X^d.

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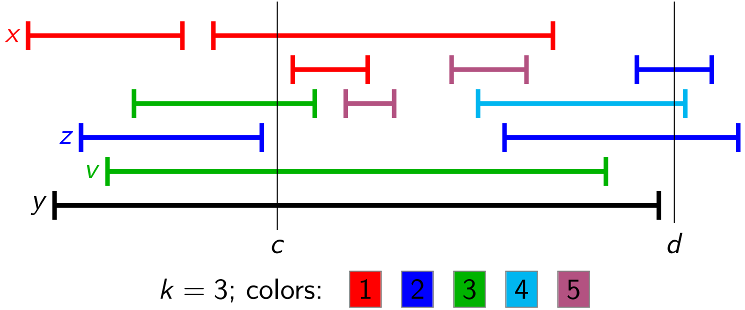

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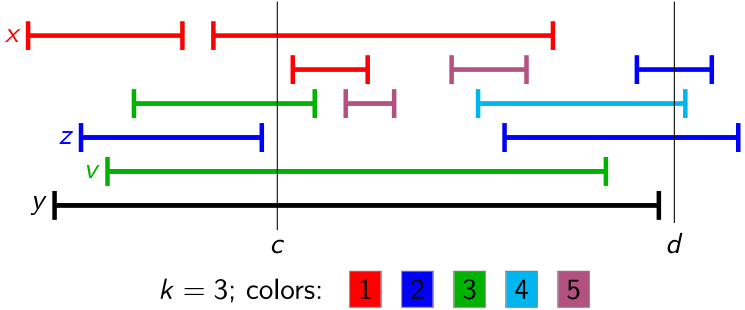

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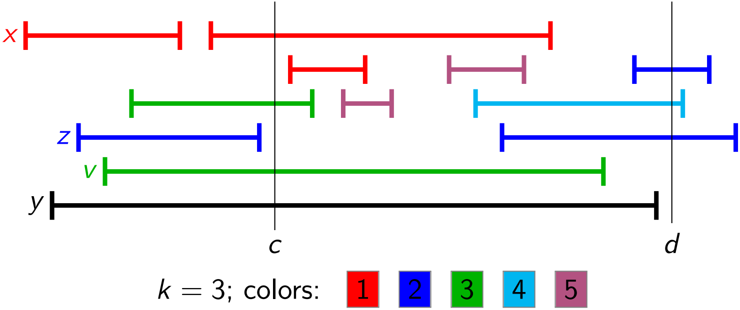

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- Swap 2 and 5 on $X^{>c}$.

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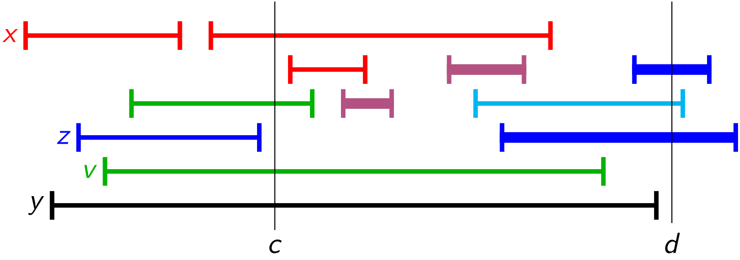

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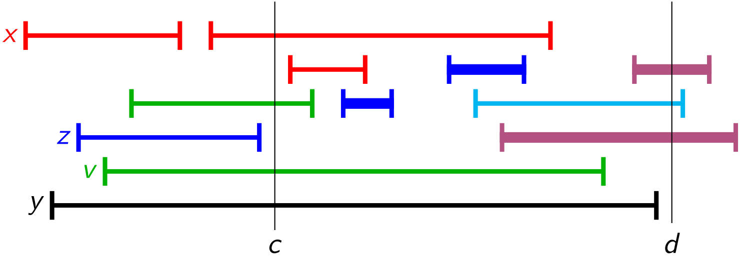
Coloring Clean Families



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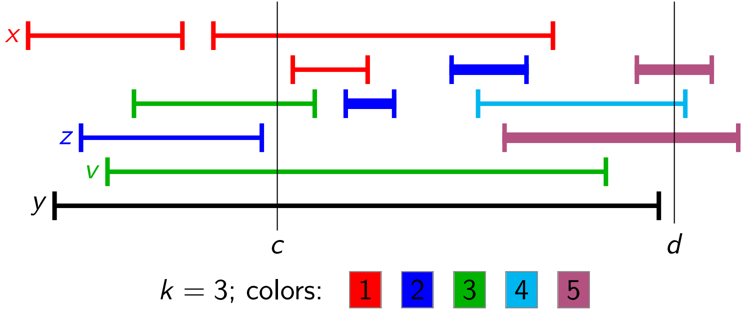

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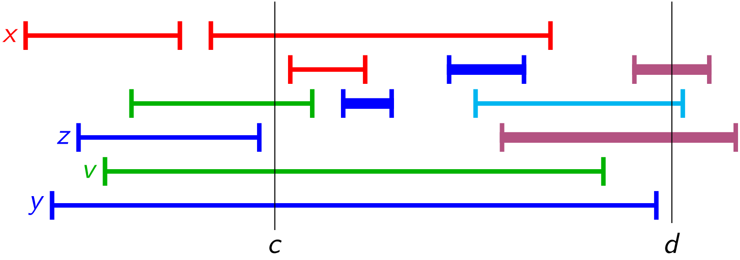

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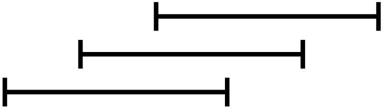

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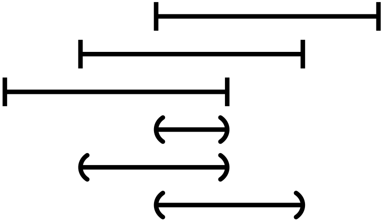

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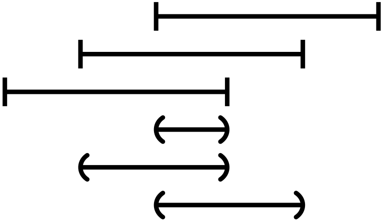

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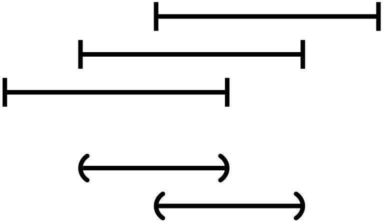

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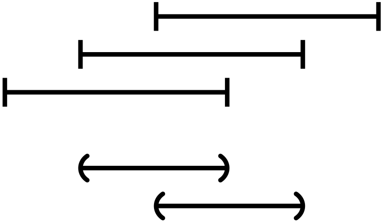

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Lemma

If X is a family of intervals whose overlap graph has clique number k, then the overlap graph of segments of X has clique number $k - 1$.

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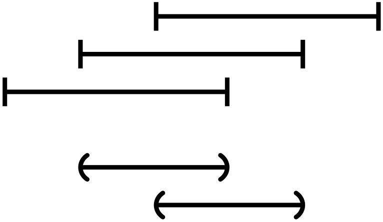
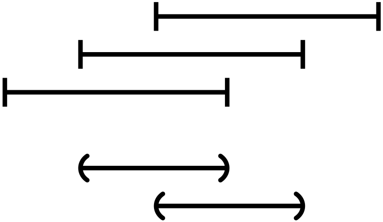


- Let X be a family of intervals.

Segments in Circle Graphs



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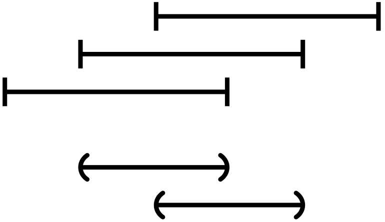

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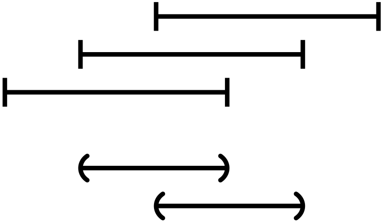

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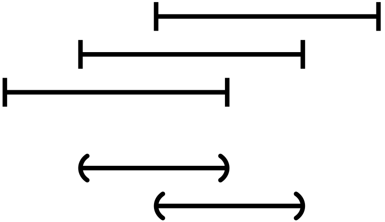

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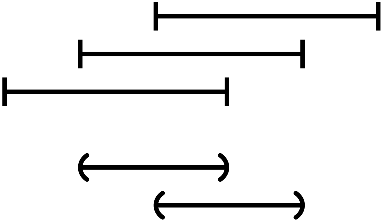

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- Using this and other tricks, we color Y.

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Thank You.