Boolean algebras and Lubell functions

Travis Johnston * Linyuan Lu † Kevin G. Milans ‡

June 7, 2015

Abstract

Let $2^{[n]}$ denote the power set of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. A collection $\mathcal{B} \subset 2^{[n]}$ forms a d-dimensional *Boolean algebra* if there exist pairwise disjoint sets $X_0, X_1, \ldots, X_d \subseteq [n]$, all non-empty with perhaps the exception of X_0, so that $\mathcal{B} = \left\{X_0 \cup \bigcup_{i \in I} X_i \colon I \subseteq [d]\right\}$. Let $b(n, d)$ be the maximum cardinality of a family $\mathcal{F} \subset 2^X$ that does not contain a d-dimensional Boolean algebra. Gunderson, Rödl, and Sidorenko proved that $b(n, d) \le c_d n^{-1/2^d} \cdot 2^n$ where $c_d = 10^d 2^{-2^{1-d}} d^{d-2^{-d}}$.

In this paper, we use the Lubell function as a new measurement for large families instead of cardinality. The Lubell value of a family of sets $\mathcal{F}$ with $\mathcal{F} \subseteq 2^{[n]}$ is defined by $h_n(\mathcal{F}) = \sum_{F \in \mathcal{F}} 1/\binom{n}{|F|}$. We prove the following Turán type theorem. If $\mathcal{F} \subseteq 2^{[n]}$ contains no d-dimensional Boolean algebra, then $h_n(\mathcal{F}) \le 2(n+1)^{1-2^{1-d}}$ for sufficiently large n. This result implies $b(n, d) \le Cn^{-1/2^d} \cdot 2^n$, where C is an absolute constant independent of n and d. With some modification, the ideas in Gunderson, Rödl, and Sidorenko's proof can be used to obtain this result. We apply the new bound on $b(n, d)$ to improve several Ramsey-type bounds on Boolean algebras. We also prove a canonical Ramsey theorem for Boolean algebras.

1 History

Given a ground set $[n]$ with $[n] = \{1, 2, \ldots, n\}$, let $2^{[n]}$ denote the power set of $[n]$.

Definition 1. *A collection $\mathcal{B} \subseteq 2^{[n]}$ forms a d-dimensional* Boolean algebra *if there exist pairwise disjoint sets $X_0, X_1, \ldots, X_d \subseteq [n]$, all non-empty with perhaps the exception of X_0, so that*

$$\mathcal{B} = \left\{ X_0 \cup \bigcup_{i \in I} X_i \colon I \subseteq [d] \right\}.$$

*University of South Carolina, Columbia, SC 29208, (johnstjt@mailbox.sc.edu).

†University of South Carolina, Columbia, SC 29208, (lu@math.sc.edu). This author was supported in part by NSF grant DMS 1300547 and ONR grant N00014-13-1-0717.

‡West Virginia University, Morgantown, WV 26506, (milans@math.wvu.edu).

We view all d-dimensional Boolean algebras as copies of a single structure $\mathcal{B}_d$. Thus, a d-dimensional Boolean algebra forms a copy of $\mathcal{B}_d$, *and a family $\mathcal{F} \subseteq 2^{[n]}$ is $\mathcal{B}_d$*-free *if it does not contain a copy of $\mathcal{B}_d$.*

The starting point of this paper is to explore the question of how large a family of sets can be if it does not contain a d-dimensional Boolean algebra. The first result in this area is due to Sperner. The simplest example of a non-trivial Boolean algebra, $\mathcal{B}_1$, is a pair of sets, one properly contained in the other. Sperner's theorem can be restated as follows. If $\mathcal{F} \subseteq 2^{[n]}$ is $\mathcal{B}_1$-free, then $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Erdős and Kleitman [4] considered the problem of determining the maximum size of a $\mathcal{B}_2$-free family in $2^{[n]}$. General extremal problems on Boolean algebras of sets were most recently studied by Gunderson, Rödl, and Sidorenko in [12].

Given an n-element set X and a positive integer d, define $b(n, d)$ to be the maximum cardinality of a $\mathcal{B}_d$-free family contained in $2^{[n]}$. In [12], the following bounds on $b(n, d)$ are proved:

$$n^{-\frac{(1+o(1))d}{2^{d+1}-2}} \cdot 2^n \leq b(n,d) \leq 10^d 2^{-2^{1-d}} d^{d-2^{-d}} n^{-1/2^d} \cdot 2^n. \tag{1}$$

In the lower bound of (1), the $o(1)$ term represents a function that tends to 0 as n grows for each fixed d. A rough overview of the proof of the upper bound on $b(n, d)$ in [12] follows. Given $\mathcal{F} \subseteq 2^{[n]}$, the set $[n]$ is partitioned into d parts $X_1, \ldots, X_d$ whose sizes differ by at most 1. Next, probabilistic techniques are used to select a family of chains $C_1, \ldots, C_d$, where each C_i is a chain in 2^{X_i} of length at least $2\lfloor\sqrt{n/d}\rfloor$. Let $\mathcal{F}_0$ be the set of all $A \in \mathcal{F}$ such that $A \cap X_i \in C_i$ for each i. The chains $C_1, \ldots, C_d$ are chosen so that $\mathcal{F}_0$ is large. Let H be the d-partite d-uniform hypergraph with parts $C_1, \ldots, C_d$ where $E(H) = \{\{A \cap X_1, \ldots, A \cap X_d\} \colon A \in \mathcal{F}_0\}$. If H contains a copy of the complete d-partite d-uniform hypergraph with 2 vertices in each part (denoted by $K^{(d)}(2, \ldots, 2)$), then $\mathcal{F}_0$ contains a d-dimensional Boolean algebra. A result of Erdős [3] implies that for n sufficiently large in terms of d, each n-vertex d-uniform hypergraph with at least $n^{d-2^{1-d}}$ edges contains a copy of $K^{(d)}(2, \ldots, 2)$. Since $|E(H)| = |\mathcal{F}_0|$ and $|\mathcal{F}_0|$ is large enough that Erdős's result applies, it follows that $\mathcal{F}$ contains a d-dimensional Boolean algebra.

With some work, the argument in [12] can be modified to eliminate the large multiplicative factor in inequality (1) that is asymptotic to $(10d)^d$. The most important modification is to exploit that H is d-partite, and in this case fewer edges force a copy of $K^{(d)}(2, \ldots, 2)$. A second, more technical modification is also necessary: the chains should be chosen to have length at least $2\lfloor\sqrt{n}/d\rfloor$. In this paper, we obtain this improvement directly, by extending a well-known result on affine cubes to Boolean algebras.

Definition 2. *A set H of integers is called a d-dimensional affine cube or an affine d-cube if there exist $d+1$ integers $x_0 \geq 0$, and $x_1, \ldots, x_d \geq 1$, such that*

$$H = \left\{ x_0 + \sum_{i \in I} x_i \colon I \subseteq [d] \right\}.$$

A set of non-negative integers is B_d-free *if it contains no affine d-cube.*

In one of the first Ramsey-type results, Hilbert [13] showed that for all d and k, there exists an integer n such that every k-coloring of $[n]$ contains a monochromatic d-dimensional affine cube. Nearly 80 years later, Szemerédi [18] strengthened Hilbert's result by proving a density version: for each positive ε and for each integer d, there exists an integer n such that if $A \subseteq [n]$ and $|A| \geq \varepsilon$, then A contains a monochromatic d-dimensional affine cube. Graham [6] strengthened Szemerédi's cube lemma by reducing the bound on $|A|$ which suffices to force a d-dimensional affine cube (see also [7]). Let $b'(n,d)$ be the maximum size of a B_d-free subset of $\{0,\ldots,n\}$. Using similar methods as in [6] and [7], problem 14.12 in [15] contains a proof that $b'(n,d) < (4(n+1))^{1-2^{1-d}}$ when n is sufficiently large in terms of d. Gunderson and Rödl [11] improved the coefficient, showing that the following holds for sufficiently large n:

$$b'(n,d) \leq 2(n+1)^{1-2^{1-d}}. \tag{2}$$

If $F \subseteq \{0,\ldots,n\}$ and $\mathcal{F} = \{A \in 2^{[n]}: |A| \in F\}$, then F contains an affine d-cube if and only if $\mathcal{F}$ contains a d-dimensional Boolean algebra. Hence, constructions that yield lower bounds on $b'(n,d)$ also yield lower bounds on $b(n,d)$. Similarly, upper bounds on $b(d,n)$ translate to upper bounds on $b'(d,n)$. The connection between large $\mathcal{B}_d$-free families in $2^{[n]}$ and large B_d-free families in $\{0,\ldots,n\}$ is simplified by using the Lubell function.

Definition 3. *Given a family $\mathcal{F} \subseteq 2^{[n]}$, we define the* Lubell function $h_n(\mathcal{F})$ *as follows:*

$$h_n(\mathcal{F}) = \sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{|F|}}.$$

With this definition in mind, we see that

$$b'(n,d) \leq \max\{h_n(\mathcal{F}) : \mathcal{F} \text{ is } \mathcal{B}_d\text{-free}\}. \tag{3}$$

The Lubell function has been widely used in the study of extremal families of sets forbidding given subposets (see [1, 8, 9, 10, 14, 17]) and in Turán problems on Non-uniform hypergraphs [16]. The advantage of using the Lubell function is its convenient probabilistic interpretation. Suppose that $\mathcal{C}$ is full-chain in $2^{[n]}$ chosen uniformly at random, i.e. $\mathcal{C} = \{\emptyset, \{i_1\}, \{i_1, i_2\}, ..., [n]\}$. Let X be the random variable $X = |\mathcal{C} \cap \mathcal{F}|$. Then we have that $\mathrm{E}(X) = h_n(\mathcal{F})$. This interpretation allows us to use tools from the probability theory (such as conditional expectation and convexity) and simplify many counting arguments.

Theorem 1. *There is a positive constant C, independent of d, such that for every d and all sufficiently large n, the following is true.*

$$b(n,d) \leq Cn^{-1/2^d} \cdot 2^n. \tag{4}$$

Our next theorem extends well-known ideas in Graham's proof of Szemerédi's cube lemma from integers to set families. The Lubell function plays a critical role and replaces cardinality as our metric for the size of a set family.

Theorem 2. *For $d \geq 1$, define $\alpha_d(n)$ recursively as follows. Let $\alpha_1(n) = 1$ and $\alpha_d(n) = \frac{1}{2} + \sqrt{2n\alpha_{d-1}(n) + \frac{1}{4}}$ for $d \geq 2$. For $n \geq d \geq 1$ if a family $\mathcal{F} \subseteq 2^{[n]}$ satisfies $h_n(\mathcal{F}) > \alpha_d(n)$, then $\mathcal{F}$ contains a d-dimensional Boolean algebra.*

The rest of the paper is organized as follows. In section 2, we prove Theorem 1 and Theorem 2. In section 3, we prove several Ramsey-type results.

2 Proofs of Theorems 1 and 2

Note that the sequence $\{\alpha_d(n)\}_{d\geq 1}$ satisfies

$$\binom{\alpha_{d+1}(n)}{2} = n\alpha_d(n) \quad \text{for } d \geq 1. \tag{5}$$

The function $\alpha_d(n)$ is used in [11] implicitly. Note that for any fixed $d \geq 2$, $\alpha_d(n)$ is an increasing function of n. We have $\alpha_1(n) = 1$, $\alpha_2(n) = \frac{1}{2} + \sqrt{2n + \frac{1}{4}}$. For $d \geq 3$, it was implicitly shown in [11] that

$$\alpha_d(n) \leq 2^{1-2^{1-d}}(\sqrt{n+1}+1)^{2-2^{2-d}} \quad \text{for } n+1 \geq 2^{d2^{d-1}/(2^{d-1}-1)}$$

and

$$\alpha_d(n) \leq 2(n+1)^{1-2^{1-d}} \quad \text{for } n+1 \geq (2^d - 2/\ln 2)^2.$$

Proof of Theorem 2:

The proof is by induction on d. For the initial case $d = 1$, we have $h_n(\mathcal{F}) > \alpha_1(n) = 1$. Let X be the number of sets in both $\mathcal{F}$ and a random full chain. Then $\mathrm{E}(X) = h_n(\mathcal{F}) > 1$. There is an instance of X satisfying $X \geq 2$. Let A and B be two sets in both $\mathcal{F}$ and a full chain. Clearly, the pair $\{A, B\}$ forms a copy of $\mathcal{B}_1$.

Assume that the statement is true for d. For $d+1$, suppose $\mathcal{F} \subseteq 2^{[n]}$ satisfies $h_n(\mathcal{F}) > \alpha_{d+1}(n)$. Let X be the number of sets in both $\mathcal{F}$ and a random full chain. By the convex inequality, we have

$$\begin{aligned}
\mathrm{E}\binom{X}{2} &\geq \binom{\mathrm{E}X}{2} \\
&> \binom{\alpha_{d+1}(n)}{2} \\
&= n\alpha_d(n).
\end{aligned}$$

For each subset S of $[n]$, let $\mathcal{F}_S = \{A \in \mathcal{F}\colon\ A \cap S = \emptyset \text{ and } A \cup S \in \mathcal{F}\}$. We show that for some non-empty set S, the Lubell function of $\mathcal{F}_S$ in $2^{[n]\setminus S}$ exceeds $\alpha_d(n - |S|)$. It follows by induction that $\mathcal{F}_S$ contains a copy of $\mathcal{B}_d$ generated by some sets $S_0, S_1, \ldots, S_d$, and with S these sets generate a copy of $\mathcal{B}_{d+1}$ in $\mathcal{F}$. Let $Z = \{(A, B) \in \mathcal{F} \times \mathcal{F}\colon\ A \subsetneq B\}$. For each $(A, B) \in Z$, the probability that a random full-chain in $2^{[n]}$ contains both A and B is $1/\binom{n}{|A|,|B|-|A|,n-|B|}$. We compute

$$\begin{aligned}
\mathrm{E}\binom{X}{2} &= \sum_{(A,B)\in Z} \frac{1}{\binom{n}{|A|,|B|-|A|,n-|B|}} \\
&= \sum_{\emptyset\subsetneq S\subseteq[n]} \sum_{A\in\mathcal{F}_S} \frac{1}{\binom{n}{|A|,|S|,n-|A|-|S|}} \\
&= \sum_{\emptyset\subsetneq S\subseteq[n]} \frac{1}{\binom{n}{|S|}} \sum_{A\in\mathcal{F}_S} \frac{1}{\binom{n-|S|}{|A|}} \\
&= \sum_{\emptyset\subsetneq S\subseteq[n]} \frac{1}{\binom{n}{|S|}} h_{n-|S|}(\mathcal{F}_S) \\
&= \sum_{k=1}^{n} \frac{1}{\binom{n}{k}} \sum_{S\in\binom{[n]}{k}} h_{n-k}(\mathcal{F}_S).
\end{aligned}$$

Since $\mathrm{E}\binom{X}{2} > n\alpha_d(n)$, it follows that $\frac{1}{\binom{n}{k}} \sum_{S\in\binom{[n]}{k}} h_{n-k}(\mathcal{F}_S) > \alpha_d(n)$ holds for some k. In turn, $h_{n-k}(\mathcal{F}_S) > \alpha_d(n) \geq \alpha_d(n-k)$ for some set S of size k. □

The following is a corollary which can be viewed as the generalization of inequality (2) and (3).

Corollary 1. *For $d \geq 3$ and $n \geq (2^d - 2/\ln 2)^2$, every family $\mathcal{F} \subseteq 2^{[n]}$ containing no d-dimensional Boolean algebra satisfies $h_n(\mathcal{F}) \leq 2(n+1)^{1-2^{1-d}}$.*

Before proving Theorem 1, we need bounds on ratios of binomial coefficients.

Lemma 1. *If $k \leq n$, then $\binom{2n}{k}/\binom{2n}{n} \leq e^{-\frac{2}{n}\binom{n-k}{2}}$.*

Proof. Note that $\binom{2n}{k}/\binom{2n}{n} = \frac{n!\cdot n!}{k!(2n-k)!} = \prod_{j=0}^{n-k-1} \frac{n-j}{n+j+1} \leq \prod_{j=0}^{n-k-1} \frac{n-j}{n+j}$. Next, we apply the inequality $(1-x)/(1+x) \leq e^{-2x}$ for $x \geq 0$ with $x = j/n$ to find $\binom{2n}{k}/\binom{2n}{n} \leq e^{-\frac{2}{n}\sum_{j=0}^{n-k-1} j} = e^{-\frac{2}{n}\binom{n-k}{2}}$. □

Proof of Theorem 1: Let $\mathcal{F} \subseteq 2^{[n]}$ be a $\mathcal{B}_d$-free family. For $0 \leq a \leq b \leq n$, let $\mathcal{F}(a,b) = \{A \in \mathcal{F}\colon a \leq |A| \leq b\}$. For two sets A and B with $A \subseteq B$, the *interval* $[A,B]$ is the set $\{X \in 2^{[n]}\colon A \subseteq X \subseteq B\}$. Let $Z = \{(A,B)\colon A \subseteq B,\ |A| = a, \text{ and } |B| = b\}$. Since $\mathcal{F}$ is $\mathcal{B}_d$-free and $[A,B]$ is a copy of the $(b-a)$-dimensional Boolean algebra, Theorem 2 implies that $h_{b-a}(\mathcal{F}\cap[A,B]) \leq \alpha_d(b-a)$ for each $(A,B) \in Z$. Since a random chain is equally likely to intersect levels a and b at all pairs in Z, it follows that $h_n(\mathcal{F}(a,b))$ is the average, over all $(A,B) \in Z$, of $h_{b-a}(\mathcal{F}\cap[A,B])$. Therefore $h_n(\mathcal{F}(a,b)) \leq \alpha_d(b-a)$.

We may assume without loss of generality that n is an even integer $2m$, and let $\ell = \lceil\sqrt{m}\rceil$. We first bound the number of sets in $\mathcal{F}$ whose size is at most m; to do this, we partition $\{A \in \mathcal{F}\colon |A| \leq m\}$ into subsets of the form $\mathcal{F}(a,b)$ where

$b-a$ is at most ℓ. Let t be the largest integer such that $m-t\ell-1 \geq 0$. We define $x_0, \ldots, x_{t+1}$ by setting $x_0 = m$, $x_j = m - j\ell - 1$ for $1 \leq j \leq t$, and $x_{t+1} = -1$. For $0 \leq j \leq t$, we define $\mathcal{F}_j = \mathcal{F}(x_{j+1}+1, x_j)$, and note that $x_j - (x_{j+1}+1) \leq \ell$ for all j. Hence $h_n(\mathcal{F}_j) \leq \alpha_d(\ell)$ for all j. Since $h_n(\mathcal{F}_j) \geq |\mathcal{F}_j|/\binom{2m}{x_j}$, it follows that $|\mathcal{F}_j| \leq \alpha_d(\ell)\binom{2m}{x_j}$.

We compute

$$\begin{aligned}\sum_{j=0}^{t} |\mathcal{F}_j| &\leq \alpha_d(\ell) \sum_{j=0}^{t} \binom{2m}{x_j} \\ &\leq \alpha_d(\ell)\binom{2m}{m} \sum_{j=0}^{t} e^{-\frac{2}{m}\binom{m-x_j}{2}} \\ &\leq \alpha_d(\ell)\binom{2m}{m} \sum_{j=0}^{t} e^{-\frac{1}{m}(j\ell)^2} \\ &\leq \alpha_d(\ell)\binom{2m}{m} \sum_{j\geq 0} e^{-\frac{\ell^2}{m}j} \\ &\leq \alpha_d(\ell)\binom{2m}{m} \frac{1}{1-e^{-\ell^2/m}},\end{aligned}$$

where we have applied Lemma 1. Since $\ell \geq \sqrt{m}$, the series is bounded by the absolute constant $1/(1-e^{-1})$. Using that $\binom{2m}{m} \leq \frac{\sqrt{2}e}{2\pi}\frac{1}{\sqrt{m}}2^{2m}$ for all m and applying our bound $\alpha_d(\ell) \leq (4\ell)^{1-2^{1-d}} \leq (4(\sqrt{m}+1))^{1-2^{1-d}} \leq 8(\sqrt{m})^{1-2^{1-d}}$ yields

$$\sum_{j=0}^{t} |\mathcal{F}_j| \leq \frac{8\sqrt{2}e^2}{2\pi(e-1)} \cdot m^{-1/2^d} \cdot 2^{2m}.$$

Doubling this, we have that $|\mathcal{F}| \leq \frac{8\sqrt{2}e^2}{\pi(e-1)} \cdot m^{-1/2^d} \cdot 2^{2m}$, and substituting $m = n/2$ gives $|\mathcal{F}| \leq \frac{16e^2}{\pi(e-1)} \cdot n^{-1/2^d} \cdot 2^n < 22n^{-1/2^d}2^n$. □

We note that our constant 22 can be reduced by sharpening the analysis in the proof of Theorem 1 in several places; we make no attempt to further reduce the constant.

3 Ramsey-type results

3.1 Multi-color Ramsey results

Given positive integers n and d, define $r(d,n)$ to be the largest integer r so that every r-coloring of $2^{[n]}$ contains a monochromatic copy of $\mathcal{B}_d$. Gunderson, Rödl, and Sidorenko [12] proved for $d > 2$,

$$cn^{1/2^d} \leq r(d,n) \leq n^{\frac{d}{2^d-1}(1+o(1))}. \tag{6}$$

Using Theorem 2, we improve the lower bound.

Theorem 3. *For $d > 2$, we have*

$$r(d,n) \geq \lfloor \frac{1}{2} n^{2/2^d} \rfloor.$$

Proof of Theorem 3: Let $r = \lfloor \frac{1}{2} n^{2/2^d} \rfloor$. For every r-coloring of $2^{[n]}$ and $1 \leq i \leq r$, let $\mathcal{F}_i$ be the family of sets in color i. By linearity, we have

$$\sum_{i=1}^{r} h_n(\mathcal{F}_i) = h_n(2^{[n]}) = n+1.$$

By the pigeonhole principle, there is a color i with $h_n(\mathcal{F}_i) \geq \frac{n+1}{r} > 2(n+1)^{1-2^{1-d}}$. For all $r, d \geq 2$, we have $n+1 \geq (2^d - 2/\ln 2)^2$. Thus,

$$h_n(\mathcal{F}_i) \geq \frac{n+1}{r} > 2(n+1)^{1-2^{1-d}} \geq \alpha_d(n).$$

By Theorem 2, $\mathcal{F}_i$ contains a copy of $\mathcal{B}_d$. $\square$

For positive integers $t_1, t_2, \ldots, t_r$, let $R(\mathcal{B}_{t_1}, \ldots, \mathcal{B}_{t_r})$ be the least integer N such that for any $n \geq N$ and any r-coloring of $2^{[n]}$ there exists an i such that $\mathcal{B}_n$ contains a monochromatic copy of $\mathcal{B}_{t_i}$ in color i. In this language, Theorem 3 states that

$$R(\underbrace{\mathcal{B}_t, \ldots, \mathcal{B}_t}_{r}) \leq (2r)^{2^{t-1}} - 1.$$

Next, we establish an exact result for $R(\mathcal{B}_s, \mathcal{B}_1)$. Our lower bound on $R(\mathcal{B}_s, \mathcal{B}_1)$ requires a numerical result. A sequence of positive integers is *complete* if every positive integer is the sum of a subsequence. In 1961, Brown [2] showed that a non-decreasing sequence $x_1, x_2, \ldots$ of positive integers with $x_1 = 1$ is complete if and only if $\sum_{i=1}^{k} x_i \leq 1 + x_{k+1}$ for each k. We adapt Brown's argument to obtain a sufficient condition for a finite variant; we include the proof for completeness.

Lemma 2 (Brown [2])**.** *Let $x_1, \ldots, x_s$ be a list of positive integers with sum at most $2s-1$. For each k with $0 \leq k \leq s$, there is a sublist with sum k.*

Proof. We use induction on s. Since the empty list of numbers has sum 0 which is larger than $2 \cdot 0 - 1$, the lemma holds vacuously when $s = 0$. For $s \geq 1$, index the integers so that $1 \leq x_1 \leq \cdots \leq x_s$. If $x_s = 1$, then $x_j = 1$ for each j and the lemma holds. Otherwise, $x_s \geq 2$ and $x_1, \ldots, x_{s-1}$ has sum at most $2(s-1)-1$. By induction, for each k with $0 \leq k \leq s-1$, some sublist of $x_1, \ldots, x_{s-1}$ has sum k. Note that $x_s \leq s$, or else $x_s \geq s+1$ and $x_j \geq 1$ for $1 \leq j \leq s-1$ would contradict that the list $x_1, \ldots, x_s$ has sum at most $2s-1$. Since $s - x_s$ is in the range $\{0, \ldots, s-1\}$, we obtain a sublist with sum s by adding x_s to a sublist of $x_1, \ldots, x_{s-1}$ with sum $s - x_s$. $\square$

Theorem 4. *For all $s \geq 1$, we have $R(\mathcal{B}_s, \mathcal{B}_1) = 2s$.*

Proof: First we show $R(\mathcal{B}_s, \mathcal{B}_1) \leq 2s$. Let $n = 2s$, let c be a red-blue coloring of $2^{[n]}$, and suppose for a contradiction that c contains neither a red copy of $\mathcal{B}_s$ nor a blue copy of $\mathcal{B}_1$. We claim that every blue set has size s. If A is blue, then all points in the up-set of A and all points in the down-set of A are red, or else the coloring has a blue copy of $\mathcal{B}_1$. If $|A| < s$, then the up-set of A contains red copies of $\mathcal{B}_s$, and if $|A| > s$, then the down-set of A contains red copies of $\mathcal{B}_s$. Therefore $|A| = s$ as claimed. Consider the copy of $\mathcal{B}_s$ generated via setting $X_0 = \emptyset$, $X_j = \{j\}$ for $1 \leq j \leq s-1$, and $X_s = \{s, s+1, \ldots, 2s\}$. None of the sets in this copy of $\mathcal{B}_s$ have size s, and therefore this is a red copy of $\mathcal{B}_s$, a contradiction.

Now we show that $R(\mathcal{B}_s, \mathcal{B}_1) > 2s - 1$. Let $n = 2s - 1$. We construct a 2-coloring of $2^{[n]}$ that contains no red copy of $\mathcal{B}_s$ and no blue copy of $\mathcal{B}_1$ as follows. Color all sets of size s blue and all other sets red. The blue sets form an antichain, so the coloring avoids blue copies of $\mathcal{B}_1$. It suffices to show that there is no red copy of $\mathcal{B}_s$. Suppose for a contradiction that a red copy of $\mathcal{B}_s$ is generated by sets $X_0, X_1, \ldots, X_s$, and let $x_j = |X_j|$. Since $X_1, \ldots, X_s$ are disjoint in $[2s-1]$ and non-empty, it follows that $x_1, \ldots, x_s$ is a list of positive integers with sum at most $2s - 1$. Since $0 \leq x_0 \leq s - 1$, we apply Lemma 2 to obtain $I \subseteq [s]$ such that $\sum_{i \in I} x_i = s - x_0$. It follows that $X_0 \cup \bigcup_{i \in I} X_i$ has size s, contradicting that the coloring contains a red copy of $\mathcal{B}_s$. □

In 1950, Erdős and Rado [5] proved the Canonical Ramsey Theorem, which lists structures that arise in every edge-coloring of the complete graph on countably many vertices. It states that each edge-coloring of the complete graph on the natural numbers contains an infinite subgraph H such that either all edges in H have the same color, or the edges in H have distinct colors, or the edges in H are colored lexicographically by their minimum or maximum endpoint. By a standard compactness argument, the Canonical Ramsey Theorem implies a finite version, stating that for each r, there is a sufficiently large n such that every edge-coloring of the complete graph on vertex set $[n]$ contains a subgraph on r vertices that is colored as in the infinitary version.

An analogous result holds for colorings of $2^{[n]}$. A coloring of $2^{[n]}$ is *rainbow* if all sets receive distinct colors. Let $\mathrm{CR}(r, s)$ be the minimum n such that every coloring of $2^{[n]}$ contains a rainbow copy of $\mathcal{B}_r$ or a monochromatic copy of $\mathcal{B}_s$. Although it is not immediately obvious that $\mathrm{CR}(r, s)$ is finite, our next theorem provides an upper bound.

Theorem 5. $\mathrm{CR}(r, s) \leq r2^{(2r+1)2^{s-1}-2}$ *for positive* r *and* s.

Proof. Set $t = 2^{(2r+1)2^{s-1}-2}$ and $n = tr$, and consider a coloring of $2^{[n]}$ that does not contain a monochromatic copy of $\mathcal{B}_s$. We obtain a rainbow copy of $\mathcal{B}_r$ with the probabilistic method. Partition the ground set $[n]$ into r sets $U_1, \ldots, U_r$ each of size t. Independently for each i in $[r]$, choose a subset X_i from U_i so that sets are chosen proportionally to their Lubell mass in the Boolean algebra on U_i. That is, each k-set in U_i has probability $\frac{1}{t+1}\binom{t}{k}^{-1}$ of being selected for X_i. For each pair $\{I, J\}$ with $I, J \subseteq [r]$, let $A_{I,J}$ be the event that both $\bigcup_{i \in I} X_i$ and $\bigcup_{j \in J} X_j$ receive the same color.

We obtain an upper bound on the probability that $A_{I,J}$ occurs. Since I and J are distinct sets, we may assume without loss of generality that there exists $m \in I - J$. Fix the selection of all sets $X_1, \ldots, X_r$ except X_m. This determines the color c of $\bigcup_{j \in J} X_j$, and the probability that $A_{I,J}$ occurs is at most the probability that $\bigcup_{i \in I} X_i$ has color c. Let L be the t-dimensional Boolean sublattice with ground set U_m, and color $B \in L$ with the same color as $B \cup \bigcup_{i \in I - \{m\}} X_i$. Let $\mathcal{F}$ be the elements in L with color c. Since $\mathcal{F}$ does not contain a monochromatic copy of $\mathcal{B}_s$, Theorem 2 implies that $h_t(\mathcal{F}) \leq (4t)^{1-2^{1-s}}$. Since

$$\begin{aligned} h_t(\mathcal{F}) &= \sum_{B \in \mathcal{F}} \binom{t}{|B|}^{-1} \\ &= (t+1) \sum_{B \in \mathcal{F}} \Pr[X_m = B] \\ &\geq (t+1) \Pr[A_{I,J}], \end{aligned}$$

we have that $\Pr[A_{I,J}] \leq (4t)^{1-2^{1-s}}/(t+1) < 4/(4t)^{2^{1-s}}$. Using the union bound, we have that the probability that at least one of the events $A_{I,J}$ occurs is less than $\binom{2^r}{2} \cdot 4/(4t)^{2^{1-s}}$, which is at most 1. It follows that for some selection of the sets $X_1, \ldots X_r$, none of the events $A_{I,J}$ occur. These sets generate a rainbow copy of $\mathcal{B}_r$. □

Note that Equation (6) implies that if $k > n^{\frac{s}{2^s-1}(1+o(1))}$, then there is a k-coloring of $2^{[n]}$ that does not contain a monochromatic copy of $\mathcal{B}_s$. Of course, with $k = 2^r - 1$, there is also no rainbow copy of $\mathcal{B}_r$. It follows that $2^r - 1 > n^{\frac{s}{2^s-1}(1+o(1))}$ implies that $\mathrm{CR}(r,s) > n$, and hence $\mathrm{CR}(r,s) \geq 2^{\frac{r(2^s-1)}{s}(1-o(1))}$ where the $o(1)$ term tends to 0 as r increases.

Acknowledgment

The authors thank the referees for helpful corrections and suggestions.

References

[1] M. Axenovich, J. Manske, and R. R. Martin, Q_2-free families in the Boolean lattice, *Order* **29** (1) (2012), 177–191.

[2] J. L. Brown, Jr., Note on complete sequences of integers, *Amer. Math. Monthly* **68** (1961), 557–560.

[3] P. Erdős, On extremal problems of graphs and generalized graphs, *Israel J. Math.* **2** (1964), 183–190.

[4] P. Erdős and D. Kleitman, On collections of subsets containing no 4-member Boolean algebra, *Proc. Amer. Math. Soc.* **28** (1971), 87–90.

[5] P. Erdős and R. Rado, A combinatorial theorem, *J. London Math. Soc.* **25** (1950), 249–255.

[6] R. L. Graham, *Rudiments of Ramsey theory*, CBMS Regional conference series in math., No. 45, AMS, 1981 (reprinted with corrections, 1983).

[7] R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey Theory*, Wiley-Interscience Ser. in Discrete Math., New York, 1990.

[8] J. R. Griggs and W.-T. Li, The partition method for poset-free families, *J. Combin. Optim.*, **25**, Issue 4, (2013), 587–596.

[9] J. Griggs, W.-T. Li, and L. Lu, Diamond-free Families, *J. Combin. Th. Ser. A*, **119** (2012) 310–322.

[10] J. Griggs and L. Lu, On families of subsets with a forbidden subposet, *Combin. Probab. Comput.*, **18** (2009) 731–748.

[11] D. S. Gunderson and V. Rödl, Extremal problems for affine cubes of integers, *Combin. Probab. and Comput.*, **7**, (1998), 65–79.

[12] D. S. Gunderson, V. Rödl, and A. Sidorenko, Extremal problems for sets forming Boolean algebras and complete partite hypergraphs, *J. Combin. Theory Ser. A*, **88**, Issue 2, (1999), 342–367.

[13] D. Hilbert, Über die Irreduzibilität ganzer rationaler Funktionen mit ganzzahligen Koeffizienten, *J. Reine Angew. Math.* **110** (1892), 104–209.

[14] L. Kramer, R. Martin, M. Young, On diamond-free subposets of the Boolean lattice, *J. Combin. Th. Ser. A*, **120**, Issue 3, (2013), 545–560.

[15] L. Lovász, *Combinatorial problems and exercises*, North-Holland Publishing Company, New York, 1979.

[16] Travis Johnston and Linyuan Lu, Turan Problems on Non-uniform Hypergraphs, *Electron. J. Combin.*, **21** (4), (2014), P22.

[17] Linyuan Lu, On crown-free families of subsets, *J. Combin. Th. Ser. A.*, **126**, (2014) 216–231.

[18] E. Szemerédi, On sets of integers containing no four elements in arithmetic progression, *Acta Math. Acad. Sci. Hungar.* **20** (1969), 199–245.